Weak Subordination of Multivariate Lévy Processes

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October 2018

A thesis submitted for the degree of Doctor of Philosophy
of the Australian National University

Declaration

The work in this thesis is my own except where otherwise stated. The results in this thesis were obtained in joint collaboration with my supervisor Boris Buchmann, while a substantial majority of Chapter 4 is my own work.

The results in this thesis have been published in the three papers entitled *Weak subordination of multivariate Lévy processes and variance generalised gamma convolutions* [BLM17], *Self-decomposability of variance generalised gamma convolutions* [BLM18b] and *Calibration for weak variance-alpha-gamma processes* [BLM18a] coauthored by Boris Buchmann, Dilip Madan and myself.

The exposition in this thesis is my own work.

Kevin Lu

Acknowledgements

I would like to thank my supervisor Assoc. Prof. Boris Buchmann. I have learned a lot from him and I am very grateful for his guidance and advice. I would also like to thank my supervisory panel, Prof. Ross Maller and Prof. Alan Welsh, for their help and support. In addition, I would like to thank Prof. Dilip Madan, Markus Michaelsen and Prof. Alexander Szimayer for their comments and discussions regarding weak subordination models and their estimation, as well as the latter two for their support and hospitality at Universität Hamburg. I also thank the three anonymous examiners for their useful comments, which have helped improve the thesis. Finally, I thank my family for their support and encouragement.

This research is supported by an Australian Government Research Training Program Scholarship, and partially supported by a Chris Heyde Scholarship and the ARC grant DP160104737.

Abstract

Based on the idea of constructing a time-changed process, strong subordination is the operation that evaluates a multivariate Lévy process at a multivariate subordinator. This produces a Lévy process again when the subordinate has independent components or the subordinator has indistinguishable components, otherwise we prove that it does not in a wide range of cases. A new operation known as *weak subordination* is introduced, acting on multivariate Lévy processes and multivariate subordinators, to extend this idea in a way that always produces a Lévy process, even when the subordinate has dependent components. We show that weak subordination matches strong subordination in law in the previously mentioned cases where the latter produces a Lévy process. In addition, we give the characteristics of weak subordination, and prove sample path properties, moment formulas and marginal component consistency. We also give distributional representations for weak subordination with ray subordinators, a superposition of independent subordinators, subordinators having independent components and subordinators having monotonic components.

The variance generalised gamma convolution class, formed by strongly subordinating Brownian motion with Thorin subordinators, is further extended using weak subordination. For these *weak variance generalised gamma convolutions*, we derive characteristics, including a formula for their Lévy measure in terms of that of a variance-gamma process, and prove sample path properties.

As an example of a weak variance generalised gamma convolution, we construct a weak subordination counterpart to the variance-alpha-gamma process of Semeraro. For these *weak variance-alpha-gamma processes*, we derive characteristics, show that they are a superposition of independent variance-gamma processes and compare three calibration methods: method of moments, maximum likelihood and digital moment estimation. As the density function is not explicitly known for maximum likelihood, we derive a Fourier invertibility condition. We show in simulations that maximum likelihood produces a better fit when this condition holds, while digital moment estimation is better when it does not. Also, weak variance-alpha-gamma processes exhibit a wider range of dependence structures and produces a significantly better fit than variance-alpha-gamma processes for the log returns of an S&P500-FTSE100 data set, and digital moment estimation has the best fit in this situation.

Lastly, we study the self-decomposability of weak variance generalised gamma convolutions. Specifically, we prove that a driftless Brownian motion gives rise to a self-decomposable process, and when some technical conditions on the underlying Thorin measure are satisfied, that this is also necessary. Our conditions improve and generalise an earlier result of Grigelionis. These conditions are applied to a variety of weakly subordinated processes, including the weak variance-alpha-gamma process, and in the previous fit, a likelihood ratio test fails to reject the self-decomposability of the log returns.

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Notation

Real and Complex Numbers

Let $\mathbb{N} = \{1, 2, 3, \dots\}$.

For $x, y \in \mathbb{R}$, let $x \wedge y := \min\{x, y\}$ and $x \vee y := \max\{x, y\}$. The decomposition of an extended real number $x \in [-\infty, \infty]$ into its positive and negative parts is denoted by $x = x^+ - x^-$, where $x^+ = x \vee 0$ and $x^- = (-x)^+ = -(x \wedge 0)$.

For $\mathbf{z} \in \mathbb{C}^n$, $\Re\mathbf{z}$ is its real part and $\Im\mathbf{z}$ is its imaginary part.

Euclidean Space

Let $\mathbb{R}^n$ be n-dimensional Euclidean space whose elements are row vectors $\mathbf{x} = (x_1, \dots, x_n)$, with canonical basis $\{\mathbf{e}_k : 1 \le k \le n\}$, and let $\mathbf{e} := (1, \dots, 1) \in \mathbb{R}^n$.

For $A \subseteq \mathbb{R}^n$, let $A_* := A \backslash \{\mathbf{0}\}$.

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\Sigma \in \mathbb{R}^{n \times n}$, let $\mathbf{x}'$ and Σ' denote the transpose of $\mathbf{x}$ and Σ, respectively.

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\Sigma \in \mathbb{R}^{n \times n}$, let $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}\mathbf{y}'$ denote the Euclidean product, $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$ denote the Euclidean norm and $\|\mathbf{x}\|_\infty = \max_{1 \le k \le n} |x_k|$ denote the infinity norm, and let $\langle \mathbf{x}, \mathbf{y} \rangle_\Sigma := \mathbf{x}\Sigma\mathbf{y}'$ and $\|\mathbf{x}\|_\Sigma^2 := \langle \mathbf{x}, \mathbf{x} \rangle_\Sigma$.

Define the Euclidean unit ball $\mathbb{D}$, the Euclidean unit sphere $\mathbb{S}$, and their associated restrictions by

$$\begin{aligned}
\mathbb{D} &:= \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \le 1\}, \\
\mathbb{D}^+ &:= \mathbb{D} \cap [0, \infty)^n, \\
\mathbb{S} &:= \{\mathbf{s} \in \mathbb{R}^n : \|\mathbf{s}\| = 1\}, \\
\mathbb{S}_+ &:= \mathbb{S} \cap [0, \infty)^n, \\
\mathbb{S}_{++} &:= \mathbb{S} \cap (0, \infty)^n, \\
\mathbb{S}^* &:= \mathbb{S} \cap (\mathbb{R}_*)^n.
\end{aligned}$$

The meaning of the notations above is understood in the usual way when used in the context of $\mathbb{R}^m$, $m \ne n$.

For $\emptyset \neq J \subseteq \{1,\dots,n\}$, let $\boldsymbol{\pi}_J : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{x} \mapsto \mathbf{x}\boldsymbol{\pi}_J := \sum_{j\in J} x_j \mathbf{e}_j$ be the associated projection.

For $\mathbf{x} = (x_1,\dots,x_n) \in \mathbb{R}^n$, let $\prod \mathbf{x} := \prod_{k=1}^n x_k$.

For $A \subseteq \mathbb{R}^n$, let ∂A denote the boundary of A relative to $\mathbb{R}^n$.

Matrices

A matrix $\Sigma \in \mathbb{R}^{n\times n}$ is a *covariance matrix*, equivalently a nonnegative definite matrix, if it is symmetric and $\|\mathbf{x}\|_\Sigma^2 \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$, and it is an *invertible covariance matrix*, equivalently a positive definite matrix, if it is symmetric and $\|\mathbf{x}\|_\Sigma^2 > 0$ for all $\mathbf{x} \in \mathbb{R}^n_*$.

Let $A = (A_{kl}) \in \mathbb{R}^{n\times n}$ and $B = (B_{kl}) \in \mathbb{R}^{n\times n}$, the Hadamard product of A and B is defined as $A * B := (A_{kl}B_{kl}) \in \mathbb{R}^{n\times n}$.

For $\Sigma \in \mathbb{R}^{n\times n}$, let $\operatorname{diag}(\Sigma) \in \mathbb{R}^n$ denote the diagonal of Σ. For $\mathbf{x} \in \mathbb{R}^n$, let $\operatorname{diag}(\mathbf{x}) \in \mathbb{R}^{n\times n}$ denote the diagonal matrix with diagonal $\mathbf{x}$.

Measures

Let $\boldsymbol{\delta}_\mathbf{x}$ denote the Dirac measure at $\mathbf{x} \in \mathbb{R}^n$.

For $\mathbf{x} \in \mathbb{R}^n$, let $\mathrm{d}\mathbf{x}$ denote the Lebesgue measure on $\mathbb{R}^n$, and $\mathrm{d}\mathbf{s}$ denote the $(n-1)$-dimensional Lebesgue surface measure on $\mathbb{S}$.

Let $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$. If $f : A \to B$ is a measurable function and $\mathcal{V}$ is a Borel measure on A, then $\mathcal{V} \circ f^{-1}$ denotes the image measure of $\mathcal{V}$ under f.

If $\mathcal{X}$ is a Borel measure on $\mathbb{R}^n_*$ and $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $\mathcal{X} \circ A^{-1}$ denotes the Borel measure on $\mathbb{R}^m_*$ constructed in the following way: extend $\mathcal{X}$ to a Borel measure $\mathcal{V}$ on $\mathbb{R}^n$ by setting $\mathcal{V}(\{\mathbf{0}\}) := 0$, and then let $\mathcal{X} \circ A^{-1}$ be the restriction of $\mathcal{V} \circ A^{-1}$ to $\mathbb{R}^m_*$.

For $\emptyset \neq J \subseteq \{1,\dots,n\}$, let $\mathcal{X}_J := \mathcal{X} \circ \boldsymbol{\pi}_J^{-1}$ be defined as above. If $J = \emptyset$, we use the conventions $\boldsymbol{\pi}_\emptyset \equiv \mathbf{0}$, $\mathcal{V}_\emptyset \equiv 0$ and $\mathcal{X}_\emptyset \equiv 0$.

Functions

Let $\mathbf{1}_A$ denote the indicator function of $A \subseteq \mathbb{R}^n$. Let $I : [0,\infty) \to [0,\infty)$ and $\ln : \mathbb{C}\backslash(-\infty, 0] \to \mathbb{C}$ denote the identity function and the principal branch of the logarithm, respectively.

A function $f : \mathbb{R} \to \mathbb{R}$ is nonincreasing if $f(x) \geq f(y)$ for all $x < y$, and it is decreasing if $f(x) > f(y)$ for all $x < y$.

For $p > 0$, L^p is the space of Borel functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $\int_{\mathbb{R}^n} |f(\mathbf{x})|^p \,\mathrm{d}\mathbf{x} < \infty$.

Random Vectors and Stochastic Processes

An n-dimensional random vector is zero if its probability distribution is $\delta_{\mathbf{0}}$. An n-dimensional process $\mathbf{X}$ is the zero process if $\mathbf{X}(t)$ is zero for all $t \geq 0$.

For n-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$, let $\mathbf{X} \stackrel{D}{=} \mathbf{Y}$ denote that $\mathbf{X}$ and $\mathbf{Y}$ are equal in distribution. For n-dimensional processes $\mathbf{X}$ and $\mathbf{Y}$, let $\mathbf{X} \stackrel{D}{=} \mathbf{Y}$ denote that $\mathbf{X}$ and $\mathbf{Y}$ are equal in law, that is their system of finite dimensional distributions are equal.

For $\boldsymbol{\mu} \in \mathbb{R}^n$ and a covariance matrix $\Sigma \in \mathbb{R}^{n\times n}$, $N(\boldsymbol{\mu}, \Sigma)$ denotes the multivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance Σ.

For $a, b > 0$, $\Gamma(a, b)$ denotes the gamma distribution with shape a and rate b.

For a process $\mathbf{X}$, the jump process $\Delta\mathbf{X}$ is defined by $\Delta\mathbf{X}(t) := \mathbf{X}(t) - \mathbf{X}(t-)$, $t > 0$, where $\mathbf{X}(t-) := \lim_{s\uparrow t} \mathbf{X}(s)$.

Let C be the name of a class of processes and P be a list of parameters. When we write $\mathbf{X} \sim C^n(P)$ without specifying the domain of the parameters P, that means we define the parameters P in the most general domain for the class of n-dimensional processes in C.

Abbreviations

We use the following abbreviations.

a.e.	almost everywhere
a.s.	almost surely
iid	independent and identically distributed
DME	digital moment estimation
GGC	generalised gamma convolution
KS	Kolmogorov-Smirnov
ML	maximum likelihood
MOM	method of moment
SD	self-decomposable
VG	variance-gamma
VGG^n	weak variance generalised gamma convolution
$VGG^{n,1}$	variance univariate generalised gamma convolution
$VGG^{n,n}$	variance multivariate generalised gamma convolution
VAG	variance-alpha-gamma
$WVAG$	weak variance-alpha-gamma

Introduction

We study the weak subordination of multivariate Lévy processes, which is a distributional extension of strong subordination that always produces a Lévy process. We apply weak subordination to generalise the class of variance generalised gamma convolutions and to construct weak variance-alpha-gamma processes, the latter allowing for more flexible dependence modelling than the corresponding strongly subordinated process. Lastly, we prove conditions for the self-decomposability of weak variance generalised gamma convolutions.

Strong Subordination

Let $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{T} = (T_1, \ldots, T_n)$ be *independent* n-dimensional Lévy processes, where $\mathbf{T}$ has nondecreasing components. *Strong subordination* is the operation that produces the time-changed process $X \circ T$ defined by

$$(\mathbf{X} \circ \mathbf{T})(t) := (X_1(T_1(t)), \ldots, X_n(T_n(t))), \quad t \geq 0.$$

We call $\mathbf{X}$ the *subordinate* and $\mathbf{T}$ the *subordinator*.

There are two important cases where it is known that $\mathbf{X} \circ \mathbf{T}$ is also a Lévy process:

(i) *univariate subordination*, where $\mathbf{T}$ has indistinguishable components;

(ii) *multivariate subordination*, where $\mathbf{X}$ has independent components.

Subordination originated with the work of Bochner [Boc55] in the context of probability transition semigroups. Univariate subordination was studied by Zolotarev [Zol58], Rogozin [Rog65] and Feller [Fel71], where the subordinate is univariate. Modern treatments of the subject can be found in the monographs by Sato [Sat99], which includes the more general situation where the subordinate is multivariate, and Barndorff-Nielsen and Shiryaev [BNS10].

More recently, multivariate subordination has been studied by Barndorff-Nielsen, Pedersen and Sato [BNPS01], who showed that this operation produces Lévy processes and derived its characteristics.

In quantitative finance, subordination acts as a time change that models the flow of information, measuring time in volume of trade or *business time* as opposed to real time. This idea was initiated by Madan and Seneta [MS90] who introduced *variance-gamma processes* (VG) for modelling stock prices and option pricing [MCC98]. A VG process is a Lévy process of the form $\mathbf{B} \circ (G\mathbf{e})$, where $\mathbf{B}$ is an n-dimensional Brownian motion, G is a univariate gamma subordinator and $\mathbf{e} = (1, \ldots, 1) \in \mathbb{R}^n$. This is an example of univariate subordination.

Typically, we want the subordinator to have both common and idiosyncratic time changes to accord with the economic intuition that some factors affect all components of the multivariate process, while others are localised to one component. However, VG processes have a time change common to all components but no idiosyncratic time changes.

This deficiency was addressed by Semeraro [Sem08] by using an *alpha-gamma* subordinator (AG), which is an application of the double gamma distribution in Kotz and Johnson [KJ72]. This subordinator was formed by the superposition of a univariate gamma subordinator that affects all components of the process to represent a common time change, and univariate gamma subordinators independently affecting each component of the process to represent idiosyncratic time changes. A *variance-alpha-gamma process* (VAG) is a Lévy process of the form $\mathbf{B} \circ \mathbf{T}$, where $\mathbf{B}$ is an n-dimensional Brownian motion with *independent* components, and $\mathbf{T}$ is an n-dimensional AG subordinator. This is an example of multivariate subordination. But the dependence structure is still restricted by requiring the Brownian motion to have independent components.

Other applications of subordination include turbulence modelling. Velocity in wind fields exhibit semi-heavy tails, symmetry, and intermittency, which gives rise to models using a univariate normal inverse Gaussian process [BN97, BN98]. Multivariate extensions also exist.

An overview of Lévy processes and strong subordination is given in Chapter 1. We show that $\mathbf{X} \circ \mathbf{T}$ can fail to be a Lévy process when the assumptions (i) and (ii) above do not hold. While there is not yet a complete characterisation of the necessary and sufficient conditions for $\mathbf{X} \circ \mathbf{T}$ to be a Lévy process, we show that these assumptions are necessary in a wide range of cases (Proposition 1.3.6). So based on the preceding discussion, strong subordination does not always create a Lévy process, and often imposes a restrictive dependence structure if we wish the result to remain a Lévy process. Of course there are numerous reasons to work in the framework of Lévy processes. In particular, they form an important subclass of semimartingales with characteristics that are deterministic and linear in time, allowing them to provide a good approximation for a wide range of random phenomena, backed up by an extensive theory [App09, Ber96, Sat99]. Furthermore, financial applications of Lévy processes are also well-developed [CS09, CT04].

Weak Subordination

Motivated by the desire to create an analogous operation to strong subordination that always produces a Lévy process and to allow for more flexible multivariate dependence modelling that improves on the VG and VAG processes, we introduce a new operation known as *weak subordination*, and its theory is developed in Chapter 2.

For a general n-dimensional Lévy process $\mathbf{X}$, *not* required to have independent components, and an n-dimensional subordinator $\mathbf{T}$, weak subordination is constructed based on the idea of assigning the distribution of $\mathbf{X}(\mathbf{t})$ to the subordinated process conditional on the subordinator taking the value $\mathbf{T}(t) = \mathbf{t}$ at time $t \geq 0$. This results in a weakly subordinated Lévy process we denote by $\mathbf{X} \odot \mathbf{T}$, whose existence will be proven (Theorem 2.2.4). Weak subordination always produces a Lévy process, and when the assumptions (i) or (ii) above are satisfied, weak subordination coincides with strong subordination in law (Theorem 2.3.5). In this sense, it is an extension of strong subordination.

For weak subordination, we derive characteristics (Section 2.3.1), marginal component consistency (Proposition 2.3.7), sample path properties (Proposition 2.3.21) and moment formulas (Proposition 2.3.22). We also give results for weak subordination in the case of a superposition of independent univariate subordinators travelling along rays (Section 2.3.4). This is a model for common and idiosyncratic time change, and our results allow for the law of weakly subordinated processes to be easily determined and understood in this situation. In addition, we show that when the subordinator has independent components, the weakly subordinated process does too (Proposition 2.3.18). There are also differences between strong and weak subordination. For instance, the time marginals of the weakly subordinated process $\mathbf{X} \odot \mathbf{T}(t)$, $t \geq 0$, coincide with that of the strongly subordinated process $\mathbf{X} \circ \mathbf{T}(t)$ when $\mathbf{T}$ is assumed to have monotonic components (Proposition 2.3.26), but not in general. In fact, there may be no Lévy process with time marginals that match $\mathbf{X} \circ \mathbf{T}(t)$ in distribution for all $t \geq 0$ (Proposition 2.3.29).

Weak Variance Generalised Gamma Convolutions

Our first major application of weak subordination will be to construct the multivariate class of weak variance generalised gamma convolutions by weakly subordinating

Brownian motion and Thorin subordinators in Chapter 3.

Generalised gamma convolutions (GGC) were introduced by Thorin [Tho77a, Tho77b] as a technical tool to prove that the lognormal and Pareto distributions were infinitely divisible by showing they were in the GGC subclass, a fact with immediate applications in modelling insurance claims. Since GGC distributions are infinitely divisible, their associated Lévy processes exist and are known as *Thorin subordinators*. In the univariate setting, examples include the gamma subordinator, the generalised inverse Gaussian subordinator [Hal79] and the $CGMY$ subordinator [JZ11].

There are several extensions of the GGC class to the multivariate setting [BNMS06, Bon09, Gri07a]. Here, as introduced by Pérez-Abreu and Stelzer [PAS14], we define the GGC class on the cone $[0,\infty)^n$ to be the minimal class of random vectors of the form $G\boldsymbol{\alpha}$, where G is a gamma random variable and $\boldsymbol{\alpha} \in [0,\infty)^n$, while being closed under convolution and convergence in distribution. The associated Thorin subordinators can be parametrised in terms of a drift and a Thorin measure, and examples include AG subordinators.

A detailed study of GGC distributions and Thorin subordinators is given in the monograph by Bondesson [Bon92]. The survey from James, Roynette and Yor [JRY08] summarises their properties and provides examples of Thorin subordinators. Both GGC distributions and Thorin subordinators have a variety of applications in the theory of infinite divisibility [BB17, BNMS06, JS13, SvH04], the analysis of Bernstein functions [SSV10], quantum probability [BNT06], and multivariate subordination models in quantitative finance [BKMS17].

Using univariate subordination, Grigelionis [Gri07b] introduced the class of processes of the form $\mathbf{B} \circ (T\mathbf{e})$, where $\mathbf{B}$ is an n-dimensional Brownian motion and T is a univariate Thorin subordinator, and this was named the $VGG^{n,1}$ class in [BKMS17]. It includes VG processes, generalised hyperbolic processes [BK01, Ebe01], $CGMY$ processes [MY08], among others.

Analogously, Buchmann et al. [BKMS17] introduced the $VGG^{n,n}$ class. Using multivariate subordination, this is the class of processes of the form $\mathbf{B} \circ \mathbf{T}$, where $\mathbf{B}$ is an n-dimensional Brownian motion with independent components, and $\mathbf{T}$ is an n-dimensional Thorin subordinator. It includes the VAG process, and all the multivariate subordination models of Luciano and Semeraro [LS10].

We use weak subordination to unify these two classes into a larger class of *weak variance generalised gamma convolutions* (VGG^n), defined as processes of the form $\mathbf{B} \odot \mathbf{T}$, where $\mathbf{B}$ is an n-dimensional Brownian motion, *not* required to have independent components, and $\mathbf{T}$ is an n-dimensional Thorin subordinator. Thus,

$VGG^n \supseteq VGG^{n,1} \cup VGG^{n,n}$.

We derive the characteristics of VGG^n processes, including a formula for their Lévy measure in terms of that of a VG process (Theorem 3.2.6), as well as proving sample path properties (Proposition 3.3.1).

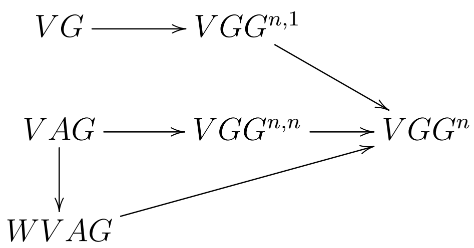

Figure 1: The relations between classes of weakly subordinated processes with arrows pointing in the direction of generalisation.

Weak Variance-Alpha-Gamma Processes

Our second major application is to construct the weak variance-alpha-gamma process to allow for more flexible multivariate dependence modelling, improving on the VG and VAG processes.

In Chapter 4, we study *weak variance-alpha-gamma processes* ($WVAG$), which are Lévy processes of the form $\mathbf{B} \odot \mathbf{T}$, where $\mathbf{B}$ is an n-dimensional Brownian motion, not required to have independent components, and $\mathbf{T}$ is an n-dimensional AG subordinator.

In particular, $WVAG$ processes are VGG^n processes, and we derive its characteristics (Proposition 4.2.2). We show that it has both common and idiosyncratic time changes with jumps caused by its AG subordinator. In addition, $WVAG$ processes have VG marginals (Proposition 4.3.1), and their moment formulas (Proposition 4.3.3) indicate that they exhibit a wider range of dependence structures than VAG processes, while still remaining parsimoniously parametrised. Moreover, a $WVAG$ process decomposes into a superposition of independent VG processes (Proposition 4.4.2), a fact that is useful for simulation.

In Section 4.6, we study calibration for $WVAG$ processes. Maximum likelihood (ML) estimation has been used to fit a univariate VG process to financial data in Madan, Carr and Chang [MCC98] and Finlay and Seneta [FS08], to fit a bivariate VG process in Fung and Seneta [FS10], to fit a $WVAG$ process in Michaelsen and Szimayer [MS18], and to fit a factor-based subordinated Brownian motion, a generalisation of the $WVAG$ process, in Wang [Wan09], Luciano, Marena and

Semeraro [LMS16], and Michaelsen and Szimayer [MS18]. Since the density function of the VAG and $WVAG$ distribution is not explicitly known but its characteristic function is, the density function is computed using Fourier inversion.

We derive a sufficient condition for Fourier invertibility of the $WVAG$ distribution (Proposition 4.5.4) in terms of its parameters, a problem which to our knowledge has not been addressed in the literature, and then we compare method of moments, the ML method of Michaelsen and Szimayer [MS18] and a modification of digital moment estimation (DME) from Madan [Mad15] in the bivariate setting. To this end, we use three goodness of fit statistics: the negative log-likelihood, a chi-squared statistic computed using the Rosenblatt transform [Ros52] and Peacock's 2-dimensional, two-sample Kolmogorov-Smirnov statistic [Pea83], the latter not requiring Fourier inversion to compute.

Using simulations, we find that ML produces a better fit when the Fourier invertibility condition is satisfied, and that DME is better when it is violated, but MLE is still surprisingly accurate in the latter case. In addition, we fit both a $WVAG$ and VAG model to the log returns from an S&P500-FTSE100 data set and show that the $WVAG$ model fits significantly better, and that DME is the better parameter estimation method in this situation.

Self-Decomposability of Weak Variance Generalised Gamma Convolutions

Originally introduced by Lévy [Lév54], self-decomposability was studied in the multivariate setting by Urbanik [Urb69], who characterised their distributions in terms of a Lévy-Khintchine representation, while Sato [Sat80] derived a criterion often used to prove self-decomposability in terms of a representation of the Lévy density in polar coordinates.

Self-decomposable distributions occur as limits of scaled sums of independent random vectors, assuming an asymptotic negligibility condition (Theorem 15.3 in [Sat99]). They also characterise the stationary distributions of multivariate Lévy-driven Ornstein-Uhlenbeck processes (Theorem 17.5 in [Sat99]).

For these reasons, self-decomposable distributions are often used to model log returns [Bin06, BK02, CGMY07, Mad18] and stochastic volatility [BNS01].

With the above insights, the question of whether the self-decomposability of a subordinator is inherited by the subordinated process has practical importance and has been the subject of considerable research. This motivates our study of the self-decomposability of VGG^n processes in Chapter 5.

Let $n = 1$ and suppose that the Brownian motion subordinate B has drift μ.

Halgreen [Hal79] proved that if T is a self-decomposable subordinator and $\mu = 0$, then $B \circ T$ is also self-decomposable. If $\mu \neq 0$, then $B \circ T$ is still self-decomposable when T is a Thorin subordinator, a particular case of a self-decomposable subordinator. This shows that all VGG^n processes are self-decomposable in the univariate case. In fact, Sato [Sat01] showed a slightly stronger result for univariate processes, that $B \circ T$ is self-decomposable when T is a self-decomposable subordinator, regardless of the value of $\mu \in \mathbb{R}$. However, this does not always hold in the multivariate case.

Now let $n \geq 2$ and suppose that the Brownian motion subordinate has drift $\boldsymbol{\mu}$. Grigelionis [Gri07b] proved that a $VGG^{n,1}$ process is self-decomposable when $\boldsymbol{\mu} = \mathbf{0}$. If some technical assumptions and a moment condition on the Thorin measure are satisfied, it is not self-decomposable when $\boldsymbol{\mu} \neq \mathbf{0}$.

More generally, for $n \geq 2$, we prove analogous conditions in the context of VGG^n processes. In particular, we show that the sufficient condition is the same (Theorem 5.2.2), and prove necessary conditions assuming weaker moment conditions than Grigelionis (Theorem 5.3.3). We apply these results to refine the self-decomposability conditions for the $VGG^{n,1}$ class, and to derive self-decomposability conditions for the $VGG^{n,n}$ class and other weakly subordinated processes (Section 5.5).

For the $WVAG$ process, the self-decomposability condition reduces to a simple criterion. Assuming that the Brownian motion subordinate has an invertible covariance matrix, self-decomposability is equivalent to $\boldsymbol{\mu} = \mathbf{0}$ (Corollary 5.5.4). Based on fitting a $WVAG$ process to the S&P500-FTSE100 data set, a likelihood ratio test fails to reject the self-decomposability of the log returns (Example 5.5.5).

Relatedly, there are two prominent generalisations of self-decomposability which were introduced by Urbanik. These are operator self-decomposability [Urb72a] and the L classes of nested operator self-decomposable distributions [Urb72b], which were further studied in [Sat80, SY85]. Operator self-decomposability allows for the previously mentioned correspondence with Lévy-driven Ornstein-Uhlenbeck processes to be generalised to matrix-valued coefficients. Barndorff-Nielsen, Pedersen and Sato [BNPS01] obtained sufficient conditions for processes formed by multivariate subordination to be included in these classes.

Structure

The text is structured as follows. Chapter 1 recaps Lévy process preliminaries and derives conditions under which strong subordination does not produce a Lévy process. Chapter 2 introduces and develops the theory of weak subordination. In Chapter 3, we use weak subordination to construct VGG^n processes. We derive their characteristics and sample path properties. In Chapter 4, we construct $WVAG$

processes, study its properties and methods for calibration. Chapter 5 studies the self-decomposability of VGG^n processes. The conclusion summarises our results and suggests possible directions for future research. A variety of useful Bessel function properties and other miscellaneous results is in Appendix A and the calibration code is in Appendix B.

Chapter 1

Strong Subordination

This chapter gives the preliminaries we will need regarding multivariate Lévy processes and the strong subordination of Lévy processes.

In Section 1.1, we review the theory of Lévy processes, characterise their laws using the Lévy-Khintchine formula and summarise important properties. In Section 1.2, we provide a new result showing that a Lévy process evaluated at a multivariate time parameter gives rise to an infinitely divisible distribution. This is an important technical result with applications throughout. In Section 1.3, we discuss strong subordination. It is well-known that strong subordination produces a Lévy process when the subordinate has independent components or the subordinator has indistinguishable components. We complement this by proving that strong subordination is not closed in the sense that it fails to produce a Lévy process in a wide range of cases where these conditions are not satisfied.

1.1 Lévy Processes

Lévy processes are the main subject of our study, and this section provides an overview of this theory. The results are primarily drawn from the monographs [App09, Ber96, Sat99], where additional information can be found.

Definition 1.1.1. A *Lévy process* $\mathbf{X} = (X_1, \ldots, X_n) = (\mathbf{X}(t))_{t\geq 0}$ is an n-dimensional stochastic process with $\mathbf{X}(0) = 0$ a.s., having independent and stationary increments, that is continuous in probability and having cádlág sample paths a.s.

Definition 1.1.2. An n-dimensional random vector $\mathbf{X}$ is *infinitely divisible* if, for all $m \geq 1$, there exist iid random vectors $\mathbf{X}_1, \ldots, \mathbf{X}_m$ such that $\mathbf{X} \overset{D}{=} \mathbf{X}_1 + \cdots + \mathbf{X}_m$.

Definition 1.1.3. The *characteristic function* of an n-dimensional random vector $\mathbf{X}$ is $\Phi_{\mathbf{X}}(\boldsymbol{\theta}) = \mathbb{E} \exp(\mathrm{i}\langle \boldsymbol{\theta}, \mathbf{X}\rangle)$, where $\boldsymbol{\theta} \in \mathbb{R}^n$.

The next two propositions connect these definitions. Specifically, there is a one-to-one correspondence between the laws of Lévy processes, infinitely divisible distributions, characteristic functions of infinitely divisible distributions, characteristic exponents and characteristic triplets, with the latter two objects being defined below. Any one of these can be used to completely and uniquely characterise the law of a Lévy process.

Proposition 1.1.4. *If* $\mathbf{X}$ *is an* n*-dimensional Lévy process, then* $\mathbf{X}(t)$*,* $t \geq 0$*, is infinitely divisible. If* $\mathbf{Y}$ *is an infinitely divisible* n*-dimensional random vector, then there exists a Lévy process* $\mathbf{X}$*, unique up to law, such that* $\mathbf{X}(1) \stackrel{D}{=} \mathbf{Y}$*.*

Proof. See Theorem 7.10 in [Sat99]. □

Recall that $\|\boldsymbol{\theta}\|_\Sigma^2 := \boldsymbol{\theta}\Sigma\boldsymbol{\theta}'$ for $\boldsymbol{\theta} \in \mathbb{R}^n$, $\Sigma \in \mathbb{R}^{n\times n}$, $\mathbb{D} := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq 1\}$ and $A_* := A\backslash\{\mathbf{0}\}$ for $A \subseteq \mathbb{R}^n$.

Proposition 1.1.5. *Let* $\mathbf{X}$ *be an* n*-dimensional Lévy process. The law of* $\mathbf{X}$ *is determined by the characteristic function* $\Phi_{\mathbf{X}} := \Phi_{\mathbf{X}(1)}$ *with*

$$\Phi_{\mathbf{X}(t)}(\boldsymbol{\theta}) = \mathbb{E}[\exp(\mathrm{i}\langle\boldsymbol{\theta}, \mathbf{X}(t)\rangle)] = \exp(t\Psi_{\mathbf{X}}(\boldsymbol{\theta})), \quad \boldsymbol{\theta} \in \mathbb{R}^n, \quad t \geq 0, \tag{1.1.1}$$

where

$$\Psi_{\mathbf{X}}(\boldsymbol{\theta}) = \mathrm{i}\langle\boldsymbol{\mu}, \boldsymbol{\theta}\rangle - \frac{1}{2}\|\boldsymbol{\theta}\|_\Sigma^2 + \int_{\mathbb{R}_*^n} (e^{\mathrm{i}\langle\boldsymbol{\theta},\mathbf{x}\rangle} - 1 - \mathrm{i}\langle\boldsymbol{\theta}, \mathbf{x}\rangle\mathbf{1}_{\mathbb{D}}(\mathbf{x}))\, \mathcal{X}(\mathrm{d}\mathbf{x}), \tag{1.1.2}$$

for some $\boldsymbol{\mu} \in \mathbb{R}^n$*, covariance matrix* $\Sigma \in \mathbb{R}^{n\times n}$ *and nonnegative Borel measure* $\mathcal{X}$ *on* $\mathbb{R}_*^n$ *satisfying*

$$\int_{\mathbb{R}_*^n} 1 \wedge \|\mathbf{x}\|^2\, \mathcal{X}(\mathrm{d}\mathbf{x}) < \infty. \tag{1.1.3}$$

Conversely, for any triplet $(\boldsymbol{\mu}, \Sigma, \mathcal{X})$*, where* $\boldsymbol{\mu} \in \mathbb{R}^n$*,* $\Sigma \in \mathbb{R}^{n\times n}$ *is a covariance matrix and* $\mathcal{X}$ *is a nonnegative Borel measure on* $\mathbb{R}_*^n$ *satisfying (1.1.3), there exists a Lévy process* $\mathbf{X}$*, unique up to law, satisfying (1.1.1) and (1.1.2).*

Proof. See Theorems 7.10 and 8.1 in [Sat99]. □

Definition 1.1.6. In Proposition 1.1.5, $\Psi_{\mathbf{X}}$ is the *characteristic exponent*, $\mathcal{X}$ is the *Lévy measure* and $(\boldsymbol{\mu}, \Sigma, \mathcal{X})$ is the *characteristic triplet*. We write $\mathbf{X} \sim L^n(\boldsymbol{\mu}, \Sigma, \mathcal{X})$ to mean that $\mathbf{X}$ is an n-dimensional Lévy process with law determined by the characteristic triplet $(\boldsymbol{\mu}, \Sigma, \mathcal{X})$.

Definition 1.1.7. If $\mathcal{X}$ is a Lévy measure on $\mathbb{R}^n_*$ and absolutely continuous with respect to a σ-finite measure $\mathcal{L}$ on $\mathbb{R}^n_*$, then the function $(\mathrm{d}\mathcal{X}/\mathrm{d}\mathcal{L})(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n_*$, satisfying

$$\mathcal{X}(A) = \int_A \frac{\mathrm{d}\mathcal{X}}{\mathrm{d}\mathcal{L}}(\mathbf{x})\,\mathcal{L}(\mathrm{d}\mathbf{x})$$

for all Borel sets $A \subseteq \mathbb{R}^n_*$ is the *Lévy density* of $\mathcal{X}$.

Equation (1.1.2) is known as the *Lévy-Khintchine formula*, and (1.1.3) ensures the finiteness of the integral in (1.1.2) and the σ-finiteness of the Lévy measure.

Infinitely divisible distributions can also be characterised by their characteristic exponents and characteristic triplets. These are defined as those of the corresponding Lévy process as determined by Proposition 1.1.4.

The next result shows that the class of Lévy processes is closed under linear transformation. In particular, the sum of independent n-dimensional Lévy processes or the projection of a Lévy process is a Lévy process.

Proposition 1.1.8. *For a Lévy process* $\mathbf{X} \sim L^n(\boldsymbol{\mu}, \Sigma, \mathcal{X})$ *and a linear transformation* $A : \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{x} \mapsto \mathbf{x}A$, *we have* $\mathbf{X}A \sim L^m(\boldsymbol{\mu}_A, \Sigma_A, \mathcal{X}_A)$, *where*

$$\begin{aligned}
\boldsymbol{\mu}_A &= \boldsymbol{\mu}A + \int_{\mathbb{R}^n_*} \mathbf{x}A(\mathbf{1}_{\mathbb{D}}(\mathbf{x}A) - \mathbf{1}_{\mathbb{D}}(\mathbf{x}))\,\mathcal{X}(\mathrm{d}\mathbf{x}),\\
\Sigma_A &= A'\Sigma A,\\
\mathcal{X}_A &= \mathcal{X} \circ A^{-1}.
\end{aligned}$$

Proof. See Proposition 11.10 in [Sat99]. □

Formulas for the moments of a Lévy process are given in Proposition 1.1.9.

Proposition 1.1.9. *If* $\mathbf{X} \sim L^n(\boldsymbol{\mu}, \Sigma, \mathcal{X})$ *is a Lévy process, then for* $t > 0$,

$$\begin{aligned}
\frac{\mathbb{E}[X(\mathbf{t})]}{t} &= \boldsymbol{\mu} + \int_{\mathbb{D}^C} \mathbf{x}\,\mathcal{X}(\mathrm{d}\mathbf{x}),\\
\frac{\mathrm{Cov}(X(\mathbf{t}))}{t} &= \Sigma + \int_{\mathbb{R}^n_*} \mathbf{x}'\mathbf{x}\,\mathcal{X}(\mathrm{d}\mathbf{x})
\end{aligned}$$

provided the participating integrals are finite.

Proof. See Example 25.12 in [Sat99]. □

Next, we introduce finite variation processes and subordinators, the latter being nondecreasing Lévy processes used to model time change.

Definition 1.1.10. Let $\mathbf{X} \sim L^n(\boldsymbol{\mu}, \Sigma, \mathcal{X})$ be a Lévy process. Then $\mathbf{X} \sim FV^n(\mathbf{d}, \mathcal{X})$ is of *finite variation* with *drift* $\mathbf{d} := \boldsymbol{\mu} - \int_{\mathbb{D}_*} \mathbf{x}\,\mathcal{X}(\mathrm{d}\mathbf{x}) \in \mathbb{R}^n$ if a.s. the sample paths of $\mathbf{X}$ are of finite variation on every compact interval.

Proposition 1.1.11. *Let* $\mathbf{X} \sim L^n(\boldsymbol{\mu}, \Sigma, \mathcal{X})$ *be a Lévy process. Then* $\mathbf{X} \sim FV^n(\mathbf{d}, \mathcal{X})$ *if and only if* $\Sigma = 0$ *and*

$$\int_{\mathbb{R}^n_*} 1 \wedge \|\mathbf{x}\|\,\mathcal{X}(\mathrm{d}\mathbf{x}) < \infty. \tag{1.1.4}$$

Also, $\mathbf{X} \sim FV^n(\mathbf{d}, \mathcal{X})$ *if and only if* $\mathbf{X}$ *has characteristic exponent*

$$\Psi_{\mathbf{X}}(\boldsymbol{\theta}) = \mathrm{i}\langle \mathbf{d}, \boldsymbol{\theta}\rangle + \int_{\mathbb{R}^n_*} \left(e^{\mathrm{i}\langle \boldsymbol{\theta}, \mathbf{x}\rangle} - 1\right)\mathcal{X}(\mathrm{d}\mathbf{x}), \quad \boldsymbol{\theta} \in \mathbb{R}^n. \tag{1.1.5}$$

Proof. See pages 16–17 in [Ber96]. □

The condition $\int_{\mathbb{D}_*} \|\mathbf{x}\|\,\mathcal{X}(\mathrm{d}\mathbf{x}) < \infty$, which is occasionally more convenient to use, is equivalent to (1.1.4) due to (1.1.3).

Definition 1.1.12. Let $\mathbf{T} \sim L^n(\boldsymbol{\mu}, \Sigma, \mathcal{T})$ be a Lévy process. Then $\mathbf{T} \sim S^n(\mathbf{d}, \mathcal{T})$ is a *subordinator* with *drift* $\mathbf{d} := \boldsymbol{\mu} - \int_{\mathbb{D}_*} \mathbf{t}\,\mathcal{T}(\mathrm{d}\mathbf{t})$ if a.s. the sample paths of $\mathbf{T}$ are nondecreasing in all components.

It is often convenient to characterise the law of a subordinator using the Laplace transform and Laplace exponent instead of the characteristic function and characteristic exponent, which are its respective Fourier transform counterparts.

Definition 1.1.13. Let $\mathbf{T} \sim S^n(\mathbf{d}, \mathcal{T})$ be a subordinator. The *Laplace transform* of $\mathbf{T}$ is $\phi_{\mathbf{T}} := \phi_{\mathbf{T}(1)}$ with

$$\phi_{\mathbf{T}(t)}(\boldsymbol{\lambda}) = \mathbb{E}[\exp(-\langle \boldsymbol{\lambda}, \mathbf{T}(t)\rangle)] = \exp(-t\Lambda_{\mathbf{T}}(\boldsymbol{\lambda})), \quad \boldsymbol{\lambda} \in [0,\infty)^n, \quad t \geq 0,$$

and $\Lambda_{\mathbf{T}}(\boldsymbol{\lambda})$ is the *Laplace exponent.*

The domain of the Laplace exponent can be extended to $\{\mathbf{z} \in \mathbb{C}^n : \Re\mathbf{z} \in [0,\infty)^n\}$. Here, we extend the Euclidean inner product to $\mathbf{w}, \mathbf{z} \in \mathbb{C}^n$ by setting $\langle \mathbf{w}, \mathbf{z}\rangle := \sum_{k=1}^n w_k z_k$. Note that there is no conjugation.

Proposition 1.1.14. *If* $\mathbf{T} \sim S^n(\mathbf{d}, \mathcal{T})$ *is a subordinator, then* $\mathbf{T} \sim FV^n(\mathbf{d}, \mathcal{T})$, $\mathbf{d} \in [0,\infty)^n$ *and* $\mathcal{T}$ *is supported on* $[0,\infty)^n_*$. *In addition,* $\mathbf{T}$ *has characteristic exponent*

$$\Psi_{\mathbf{T}}(\boldsymbol{\theta}) = \mathrm{i}\langle \mathbf{d}, \boldsymbol{\theta}\rangle + \int_{[0,\infty)^n_*} (e^{\mathrm{i}\langle \boldsymbol{\theta}, \mathbf{t}\rangle} - 1)\,\mathcal{T}(\mathrm{d}\mathbf{t}), \quad \boldsymbol{\theta} \in \mathbb{R}^n, \tag{1.1.6}$$

and Laplace exponent

$$\Lambda_{\mathbf{T}}(\mathbf{z}) = \langle \mathbf{d}, \mathbf{z} \rangle + \int_{[0,\infty)^n_*} (1 - e^{-\langle \mathbf{z}, \mathbf{t} \rangle})\, \mathcal{T}(\mathrm{d}\mathbf{t}), \quad \Re \mathbf{z} \in [0, \infty)^n. \tag{1.1.7}$$

Proof. The first sentence and (1.1.6) follows from Proposition 1.1.11 above and Proposition 3.1 in [BNPS01]. The Laplace exponent (1.1.7) is in the proof of Theorem 3.3 in [BNPS01]. □

Next, we give some examples of Lévy processes, the most important being Brownian motion and the gamma subordinator, which will be used throughout.

Definition 1.1.15. An n-dimensional Lévy process $\mathbf{B} \sim BM^n(\boldsymbol{\mu}, \Sigma)$ is a *Brownian motion* with drift $\boldsymbol{\mu}$ and covariance Σ if $\mathbf{B} \sim L^n(\boldsymbol{\mu}, \Sigma, 0)$. If $B \sim BM^1(0, 1)$, then B is a *standard Brownian motion.*

Definition 1.1.16. Let $a, b > 0$. A univariate subordinator $G \sim \Gamma_S(a, b)$ is a *gamma subordinator* or a gamma process with shape a and rate b if $G \sim S^1(0, \mathcal{G}_{a,b})$ with Lévy mesaure

$$\mathcal{G}_{a,b}(\mathrm{d}g) := \mathbf{1}_{(0,\infty)}(g) a e^{-bg} \frac{\mathrm{d}g}{g}. \tag{1.1.8}$$

If $a = b$, then G is a *standard gamma subordinator*, and we let $\Gamma_S(b) := \Gamma_S(b, b)$, $\mathcal{G}_b := \mathcal{G}_{b,b}$.

Remark 1.1.17. Recall that $\Gamma(a, b)$, $a, b > 0$, denotes the gamma distribution with shape a and rate b. The gamma subordinator G can also be defined as the subordinator with time marginals $G(t) \sim \Gamma(at, b)$, $t \geq 0$. In particular, $\mathbb{E}[G(1)] = 1$ if and only if G is a standard gamma subordinator. Alternatively, G can be characterised using its Laplace exponent

$$\Lambda_G(\lambda) = a \ln\left(1 + \frac{\lambda}{b}\right), \quad \lambda > -b. \tag{1.1.9}$$

Definition 1.1.18. Let $\lambda > 0$ and $\mathcal{P}$ be a probability measure on $\mathbb{R}^n_*$. An n-dimensional Lévy process $\mathbf{N} \sim FV^n(\mathbf{0}, \lambda\mathcal{P})$ is a *compound Poisson process* with rate λ and jump size distribution $\mathcal{P}$. If $n = 1$ and $\mathcal{P} = \boldsymbol{\delta}_1$, then N is a *Poisson process.*

1.2 Multivariate Time Parameters

We consider the evaluation of an n-dimensional Lévy process $\mathbf{X} = (X_1, \ldots, X_n)$, indexed by a univariate time parameter $t \geq 0$, at a multivariate time parameter

$\mathbf{t} = (t_1, \ldots, t_n) \in [0, \infty)^n$. Let

$$\mathbf{X}(\mathbf{t}) := (X_1(t_1), \ldots, X_n(t_n)).$$

We will show that this is an n-dimensional infinitely divisible random vector and give its characteristics. This is an important technical tool for later results.

To provide formulas for these characteristics, we introduce an outer product operation $\diamond$. For $\mathbf{t} = (t_1, \ldots, t_n) \in [0, \infty)^n$, $\boldsymbol{\mu} = (\mu_1, \ldots \mu_n) \in \mathbb{R}^n$ and $\Sigma = (\Sigma_{kl}) \in \mathbb{R}^{n \times n}$, let $\mathbf{t} \diamond \boldsymbol{\mu} \in \mathbb{R}^n$ and $\mathbf{t} \diamond \Sigma = ((\mathbf{t} \diamond \Sigma)_{kl}) \in \mathbb{R}^{n \times n}$ be defined by

$$\mathbf{t} \diamond \boldsymbol{\mu} := (t_1\mu_1, \ldots, t_n\mu_n), \quad (\mathbf{t} \diamond \Sigma)_{kl} := (t_k \wedge t_l)\Sigma_{kl}, \quad 1 \le k, l \le n. \tag{1.2.1}$$

Choose an ordering $t_{(1)} \le \cdots \le t_{(n)}$ of the components of $\mathbf{t}$ with associated permutation $\langle (1), \ldots, (n) \rangle$, and define the spacings $\Delta t_{(k)} := t_{(k)} - t_{(k-1)}$, $1 \le k \le n$, with $t_{(0)} := 0$. Recall that for $\emptyset \ne J \subseteq \{1, \ldots, n\}$, the associated projection $\boldsymbol{\pi}_J$ is defined by $\mathbf{x}\boldsymbol{\pi}_J := \sum_{j \in J} x_j \mathbf{e}_j$, $\mathbf{x} \in \mathbb{R}^n$. Let $\mathcal{X}_J := \mathcal{X} \circ \boldsymbol{\pi}_J^{-1}$ be constructed in the usual way. That is, extend $\mathcal{X}$ to a Borel measure $\mathcal{V}$ on $\mathbb{R}^n$ by setting $\mathcal{V}(\{\mathbf{0}\}) := 0$, and then let $\mathcal{X} \circ \boldsymbol{\pi}_J^{-1}$ be the restriction of $\mathcal{V} \circ \boldsymbol{\pi}_J^{-1}$ to $\mathbb{R}^n_*$. For a Lévy measure $\mathcal{X}$, let

$$\mathbf{t} \diamond \mathcal{X} := \sum_{k=1}^{n} \Delta t_{(k)} \mathcal{X}_{\{(k), \ldots, (n)\}}. \tag{1.2.2}$$

Introduce the compensation term

$$\mathbf{c} := \mathbf{c}(\mathbf{t}, \mathcal{X}) := \sum_{k=2}^{n} \Delta t_{(k)} \int_{\mathbb{D}^C} \mathbf{x}\boldsymbol{\pi}_{\{(k), \ldots, (n)\}} \mathbf{1}_{\mathbb{D}}(\mathbf{x}\boldsymbol{\pi}_{\{(k), \ldots, (n)\}})\, \mathcal{X}(\mathrm{d}\mathbf{x}). \tag{1.2.3}$$

Note that $\mathbf{t} \diamond \mathcal{X}$ and $\mathbf{c}(\mathbf{t}, \mathcal{X})$ are well-defined since they are invariant under any permutation with the same ordering of $\mathbf{t}$. Moreover, the following lemma shows that $\mathbf{c}(\mathbf{t}, \mathcal{X}) \in \mathbb{R}^n$, $\mathbf{t} \diamond \Sigma$ is a covariance matrix whenever Σ is, and $\mathbf{t} \diamond \mathcal{X}$ is a Lévy measure.

Lemma 1.2.1. *Let* $\mathbf{t} \in [0, \infty)^n$. *If* $(\boldsymbol{\mu}, \Sigma, \mathcal{X})$ *is a characteristic triplet, then so is* $(\mathbf{t} \diamond \boldsymbol{\mu} + \mathbf{c}, \mathbf{t} \diamond \Sigma, \mathbf{t} \diamond \mathcal{X})$*, and*

$$\|\mathbf{c}(\mathbf{t}, \mathcal{X})\| \le n^{1/2} \mathcal{X}(\mathbb{D}^C) \|\mathbf{t}\|. \tag{1.2.4}$$

Proof. Note that

$$\|\mathbf{x}\|_\infty \le \|\mathbf{x}\| \le n^{1/2} \|\mathbf{x}\|_\infty, \quad \mathbf{x} \in \mathbb{R}^n, \tag{1.2.5}$$

which gives

$$\begin{aligned}\|\mathbf{c}(\mathbf{t},\mathcal{X})\|_\infty &= \sum_{k=2}^{n} \Delta t_{(k)} \int_{\mathbb{D}^C} \|\boldsymbol{\pi}_{\{(k),\ldots,(n)\}}(\mathbf{x})\|_\infty \mathbf{1}_{\mathbb{D}}(\boldsymbol{\pi}_{\{(k),\ldots,(n)\}}(\mathbf{x}))\,\mathcal{X}(\mathrm{d}\mathbf{x}) \\ &\le (t_{(n)} - t_{(1)})\mathcal{X}(\mathbb{D}^C) \\ &\le \|\mathbf{t}\|\mathcal{X}(\mathbb{D}^C).\end{aligned}$$

Since $\mathcal{X}(\mathbb{D}^C)$ is finite by (1.1.3), $\mathbf{c}(\mathbf{t},\mathcal{X}) \in \mathbb{R}^n$ and (1.2.4) follows from the above inequalities. In particular, $\mathbf{t} \diamond \boldsymbol{\mu} + \mathbf{c} \in \mathbb{R}^n$. Let $\mathbf{B} \sim BM^n(\mathbf{0},\Sigma)$, then $\mathrm{Cov}(\mathbf{B}(\mathbf{t})) = \mathbf{t} \diamond \Sigma$, so $\mathbf{t} \diamond \Sigma$ is a covariance matrix. Finally, by Proposition 1.1.8, $\mathcal{X}_{\{(k),\ldots,(n)\}}$, $1 \le k \le n$, is a Lévy measure. This implies, by (1.1.3), that $\mathbf{t} \diamond \mathcal{X}$ is also a Lévy measure. □

Proposition 1.2.2. *For* $\mathbf{t} \in [0,\infty)^n$ *and* $\mathbf{X} \sim L^n(\boldsymbol{\mu},\Sigma,\mathcal{X})$*, the random vector* $\mathbf{X}(\mathbf{t})$ *is infinitely divisible with characteristic function*

$$\Phi_{\mathbf{X}(\mathbf{t})}(\boldsymbol{\theta}) = \mathbb{E}[\exp(\mathrm{i}\langle \boldsymbol{\theta}, \mathbf{X}(\mathbf{t})\rangle)] = \exp((\mathbf{t} \diamond \Psi_{\mathbf{X}})(\boldsymbol{\theta})), \quad \boldsymbol{\theta} \in \mathbb{R}^n,$$

where

$$(\mathbf{t} \diamond \Psi_{\mathbf{X}})(\boldsymbol{\theta}) := \sum_{k=1}^{n} \Delta t_{(k)} \Psi_{\mathbf{X}}(\boldsymbol{\pi}_{\{(k),\ldots,(n)\}}(\boldsymbol{\theta})) \tag{1.2.6}$$

$$= \mathrm{i}\langle \mathbf{t} \diamond \boldsymbol{\mu} + \mathbf{c}, \boldsymbol{\theta}\rangle - \frac{1}{2}\|\boldsymbol{\theta}\|^2_{\mathbf{t}\diamond\Sigma} + \int_{\mathbb{R}^n_*} (e^{\mathrm{i}\langle\boldsymbol{\theta},\mathbf{x}\rangle} - 1 - \mathrm{i}\langle\boldsymbol{\theta},\mathbf{x}\rangle \mathbf{1}_{\mathbb{D}}(\mathbf{x}))\,(\mathbf{t} \diamond \mathcal{X})(\mathrm{d}\mathbf{x}). \tag{1.2.7}$$

Proof. Let $\boldsymbol{\pi}_m := \boldsymbol{\pi}_{\{(m),\ldots,(n)\}}$, $1 \le m \le n$. For $\mathbf{X} = (X_1,\ldots,X_n) \sim L^n(\boldsymbol{\mu},\Sigma,\mathcal{X})$, $\mathbf{t} = (t_1,\ldots,t_n) \in [0,\infty)^n$ and $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_n) \in \mathbb{R}^n$, we have

$$\langle \boldsymbol{\theta}, \mathbf{X}(\mathbf{t})\rangle = \sum_{k=1}^{n} \theta_{(k)} X_{(k)}(t_{(k)}) = \sum_{k=1}^{n}\sum_{m=1}^{k} \theta_{(k)}(X_{(k)}(t_{(m)}) - X_{(k)}(t_{(m-1)})),$$

and thus, by interchanging the order of summation on the RHS, we have

$$\langle \boldsymbol{\theta}, \mathbf{X}(\mathbf{t})\rangle = \sum_{m=1}^{n}\sum_{k=m}^{n} \theta_{(k)}(X_{(k)}(t_{(m)}) - X_{(k)}(t_{(m-1)})).$$

Combining the above equation with the independent and stationary increment property of $\mathbf{X}$ and using (1.1.1) gives

$$\mathbb{E}[\exp(\mathrm{i}\langle\boldsymbol{\theta},\mathbf{X}(\mathbf{t})\rangle)] = \exp\left(\sum_{m=1}^{n}\Delta t_{(m)}\Psi_{\mathbf{X}}(\boldsymbol{\theta}\boldsymbol{\pi}_m)\right), \tag{1.2.8}$$

which proves (1.2.6). Since projections are self-adjoint, meaning that $\langle\mathbf{x}\boldsymbol{\pi}_m,\mathbf{y}\rangle = \langle\mathbf{x},\mathbf{y}\boldsymbol{\pi}_m\rangle$, $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$, using the Lévy-Khintchine formula (1.1.2) gives

$$\begin{aligned}\Psi_{\mathbf{X}}(\boldsymbol{\theta}\boldsymbol{\pi}_m) = \mathrm{i}\langle\boldsymbol{\mu}\boldsymbol{\pi}_m,\boldsymbol{\theta}\rangle - \frac{1}{2}\|\boldsymbol{\theta}\boldsymbol{\pi}_m\|_{\Sigma}^2 \\ + \int_{\mathbb{R}_*^n}\left(e^{\mathrm{i}\langle\boldsymbol{\theta},\mathbf{x}\boldsymbol{\pi}_m\rangle} - 1 - \mathrm{i}\langle\boldsymbol{\theta},\mathbf{x}\boldsymbol{\pi}_m\rangle\mathbf{1}_{\mathbb{D}}(\mathbf{x})\right)\mathcal{X}(\mathrm{d}\mathbf{x}).\end{aligned} \tag{1.2.9}$$

Substituting (1.2.9) into (1.2.8) gives

$$\mathbb{E}[\exp(\mathrm{i}\langle\boldsymbol{\theta},\mathbf{X}(\mathbf{t})\rangle)] = \exp(I_1(\boldsymbol{\theta}) + I_2(\boldsymbol{\theta}) + I_3(\boldsymbol{\theta})), \tag{1.2.10}$$

where

$$\begin{aligned}I_1(\boldsymbol{\theta}) &:= \sum_{m=1}^{n}\Delta t_{(m)}\langle\boldsymbol{\mu}\boldsymbol{\pi}_m,\boldsymbol{\theta}\rangle,\\ I_2(\boldsymbol{\theta}) &:= \sum_{m=1}^{n}\Delta t_{(m)}\|\boldsymbol{\theta}\boldsymbol{\pi}_m\|_{\Sigma}^2,\\ I_3(\boldsymbol{\theta}) &:= \sum_{m=1}^{n}\Delta t_{(m)}\int_{\mathbb{R}_*^n}\left(e^{\mathrm{i}\langle\boldsymbol{\theta},\mathbf{x}\boldsymbol{\pi}_m\rangle} - 1 - \mathrm{i}\langle\boldsymbol{\theta},\mathbf{x}\boldsymbol{\pi}_m\rangle\mathbf{1}_{\mathbb{D}}(\mathbf{x})\right)\mathcal{X}(\mathrm{d}\mathbf{x}).\end{aligned}$$

To deal with the first term, note that

$$I_1(\boldsymbol{\theta}) = \sum_{m=1}^{n}\Delta t_{(m)}\sum_{k=m}^{n}\mu_{(k)}\theta_{(k)} = \sum_{k=1}^{n}\mu_{(k)}\theta_{(k)}\sum_{m=1}^{k}\Delta t_{(m)} = \langle\mathbf{t}\diamond\boldsymbol{\mu},\boldsymbol{\theta}\rangle,$$

and likewise the second term becomes

$$I_2(\boldsymbol{\theta}) = \sum_{m=1}^{n}\Delta t_{(m)}\sum_{k=m}^{n}\sum_{l=m}^{n}\theta_{(k)}\theta_{(l)}\Sigma_{(k)(l)} = \sum_{k=1}^{n}\sum_{l=1}^{n}\theta_{(k)}\theta_{(l)}\Sigma_{(k)(l)}\sum_{m=1}^{k\wedge l}\Delta t_{(m)} = \|\boldsymbol{\theta}\|_{\mathbf{t}\diamond\Sigma}^2.$$

By using $\mathbf{1}_{\mathbb{D}}(\mathbf{x}) = \mathbf{1}_{\mathbb{D}}(\mathbf{x}\boldsymbol{\pi}_m) - \mathbf{1}_{\mathbb{D}^C}(\mathbf{x})\mathbf{1}_{\mathbb{D}}(\mathbf{x}\boldsymbol{\pi}_m)$, $\mathbf{x}\in\mathbb{R}_*^n$, and applying the transformation theorem (see Proposition A.3.1), we get

$$\begin{aligned}I_3(\boldsymbol{\theta}) = \mathrm{i}\left\langle\boldsymbol{\theta},\sum_{m=1}^{n}\Delta t_{(m)}\int_{\mathbb{D}^C}\mathbf{x}\boldsymbol{\pi}_m\mathbf{1}_{\mathbb{D}}(\mathbf{x}\boldsymbol{\pi}_m)\,\mathcal{X}(\mathrm{d}\mathbf{x})\right\rangle\\ + \sum_{m=1}^{n}\Delta t_{(m)}\int_{\mathbb{R}_*^n}\left(e^{\mathrm{i}\langle\boldsymbol{\theta},\mathbf{x}\rangle} - 1 - \mathrm{i}\langle\boldsymbol{\theta},\mathbf{x}\rangle\mathbf{1}_{\mathbb{D}}(\mathbf{x})\right)(\mathcal{X}\circ\boldsymbol{\pi}_m^{-1})(\mathrm{d}\mathbf{x})\end{aligned}$$

$$= \mathrm{i}\langle \mathbf{c}, \boldsymbol{\theta} \rangle + \int_{\mathbb{R}^n_*} \left(e^{\mathrm{i}\langle \boldsymbol{\theta}, \mathbf{x} \rangle} - 1 - \mathrm{i}\langle \boldsymbol{\theta}, \mathbf{x} \rangle \mathbf{1}_{\mathbb{D}}(\mathbf{x}) \right) (\mathbf{t} \diamond \mathcal{X})(\mathrm{d}\mathbf{x}),$$

where the last line is obtained by recalling the definitions in (1.2.2) and (1.2.3). Finally, substituting these expressions for $I_1(\boldsymbol{\theta})$, $I_2(\boldsymbol{\theta})$ and $I_3(\boldsymbol{\theta})$ into (1.2.10) proves (1.2.7).

Note that $\mathbf{t} \diamond \Psi_{\mathbf{X}}$ is in the form of (1.1.2) with characteristic triplet $(\mathbf{t} \diamond \boldsymbol{\mu} + \mathbf{c}, \mathbf{t} \diamond \Sigma, \mathbf{t} \diamond \mathcal{X})$ in light of Lemma 1.2.1. Therefore, $\mathbf{X}(\mathbf{t})$ is infinitely divisible by Proposition 1.1.4. □

By Proposition 1.2.2, for $\mathbf{d} \in [0, \infty)^n$ and $\mathbf{X} \sim L^n(\boldsymbol{\mu}, \Sigma, \mathcal{X})$, $\mathbf{X}(\mathbf{d})$ is infinitely divisible, so it is associated with the Lévy process $\mathbf{Y} \sim L^n(\mathbf{d} \diamond \boldsymbol{\mu} + \mathbf{c}, \mathbf{d} \diamond \Sigma, \mathbf{d} \diamond \mathcal{X})$ satisfying $\mathbf{Y}(t) \stackrel{D}{=} \mathbf{X}(t\mathbf{d})$ for all $t \geq 0$.

Example 1.2.3. Let $\mathbf{B} \sim BM^n(\boldsymbol{\mu}, \Sigma)$ and $\mathbf{t} \in [0, \infty)^n$. Then

$$\mathbf{B}(\mathbf{t}) \sim N(\mathbf{t} \diamond \boldsymbol{\mu}, \mathbf{t} \diamond \Sigma). \tag{1.2.11}$$

This is an infinitely divisible random vector with characteristic exponent

$$\mathbf{t} \diamond \Psi_{\mathbf{B}}(\boldsymbol{\theta}) = \mathrm{i}\langle \mathbf{t} \diamond \boldsymbol{\mu}, \boldsymbol{\theta} \rangle - \frac{1}{2} \|\boldsymbol{\theta}\|^2_{\mathbf{t} \diamond \Sigma}, \quad \boldsymbol{\theta} \in \mathbb{R}^n,$$

so the associated Lévy process is $\mathbf{B}^{(\mathbf{t})} \sim BM^n(\mathbf{t} \diamond \boldsymbol{\mu}, \mathbf{t} \diamond \Sigma)$.

1.3 Nonclosure of Strong Subordination

This section studies strong subordination. We recall sufficient conditions for this operation to produce a Lévy process, and then we find conditions for when it does not. Due to the latter result, we say that strong subordination is not closed.

Definition 1.3.1. Let $\mathbf{X} = (X_1, \ldots, X_n) \sim L^n(\boldsymbol{\mu}, \Sigma, \mathcal{X})$ be a Lévy process and $\mathbf{T} = (T_1, \ldots, T_n) \sim S^n(\mathbf{d}, \mathcal{T})$ be a subordinator independent of $\mathbf{X}$. The process $\mathbf{X} \circ \mathbf{T}$ is the *strong subordination* of $\mathbf{X}$ and $\mathbf{T}$ if

$$(\mathbf{X} \circ \mathbf{T})(t) := (X_1(T_1(t)), \ldots, X_n(T_n(t))), \quad t \geq 0.$$

If $\mathbf{T}$ has indistinguishable components, then $\mathbf{X} \circ \mathbf{T}$ is the *univariate subordination* of $\mathbf{X}$ and $\mathbf{T}$. If $\mathbf{X}$ has independent components, then $\mathbf{X} \circ \mathbf{T}$ is the *multivariate subordination* of $\mathbf{X}$ and $\mathbf{T}$.

In the literature, the term "subordination" is often used instead of "strong subordination". We use the latter to distinguish this traditional notion of subordination from weak subordination, which is introduced in Chapter 2.

There are two special cases where strong subordination is known to produce a Lévy process. Recall that $\mathbf{e} := (1, \ldots, 1) \in \mathbb{R}^n$.

Proposition 1.3.2. *Let* $\mathbf{X} \sim L^n(\boldsymbol{\mu}, \Sigma, \mathcal{X})$ *and* $\mathbf{T} \sim S^n(\mathbf{d}, \mathcal{T})$ *be independent. If*

(i) $\mathbf{T}$ *has indistinguishable components with* $\mathbf{T} = R\mathbf{e}$, $R \sim S^1(d, \mathcal{R})$, *or*

(ii) $\mathbf{X}$ *has independent components,*

then $\mathbf{Y} \stackrel{D}{=} \mathbf{X} \circ \mathbf{T}$ *is a Lévy process.*

Under (i), $\mathbf{Y} \sim L^n(\mathbf{m}, \Theta, \mathcal{Y})$, *where*

$$\begin{aligned}
\mathbf{m} &= d\boldsymbol{\mu} + \int_{(0,\infty)} \mathbb{E}[\mathbf{X}(r)\mathbf{1}_{\mathbb{D}}(\mathbf{X}(r))]\, \mathcal{R}(\mathrm{d}r),\\
\Theta &= d\Sigma,\\
\mathcal{Y}(\mathrm{d}\mathbf{x}) &= d\mathcal{X}(\mathrm{d}\mathbf{x}) + \int_{(0,\infty)} \mathbb{P}(\mathbf{X}(r) \in \mathrm{d}\mathbf{x})\mathcal{R}(\mathrm{d}r).
\end{aligned}$$

Under (ii), $\mathbf{Y} \sim L^n(\mathbf{m}, \Theta, \mathcal{Y})$, *where*

$$\begin{aligned}
\mathbf{m} &= \mathbf{d} \diamond \boldsymbol{\mu} + \int_{[0,\infty)^n_*} \mathbb{E}[\mathbf{X}(\mathbf{t})\mathbf{1}_{\mathbb{D}}(\mathbf{X}(\mathbf{t}))]\, \mathcal{T}(\mathrm{d}\mathbf{t}),\\
\Theta &= \mathbf{d} \diamond \Sigma,\\
\mathcal{Y}(\mathrm{d}\mathbf{x}) &= \mathbf{d} \diamond \mathcal{X}(\mathrm{d}\mathbf{x}) + \int_{[0,\infty)^n_*} \mathbb{P}(\mathbf{X}(\mathbf{t}) \in \mathrm{d}\mathbf{x})\mathcal{T}(\mathrm{d}\mathbf{t}).
\end{aligned} \tag{1.3.1}$$

Proof. See Theorem 30.1 in [Sat99] and Theorem 3.3 in [BNPS01]. □

The measure $\int_{[0,\infty)^n_*} \mathbb{P}(\mathbf{X}(\mathbf{t}) \in \mathrm{d}\mathbf{x})\mathcal{T}(\mathrm{d}\mathbf{t})$ is defined by $A \mapsto \int_{[0,\infty)^n_*} \mathbb{P}(X(\mathbf{t}) \in A)\, \mathcal{T}(\mathrm{d}\mathbf{t})$ for all Borel sets $A \subseteq \mathbb{R}^n_*$.

Next, we review one of the most important applications of strong subordination, the variance-gamma process.

Definition 1.3.3. Let $b > 0$, $\boldsymbol{\mu} \in \mathbb{R}^n$ and $\Sigma \in \mathbb{R}^{n \times n}$ be a covariance matrix. An n-dimensional Lévy process $\mathbf{V} \sim VG^n(b, \boldsymbol{\mu}, \Sigma)$ is a *variance-gamma process* if $\mathbf{V} \stackrel{D}{=} \mathbf{B} \circ (G\mathbf{e})$, where $\mathbf{B} \sim BM^n(\boldsymbol{\mu}, \Sigma)$ and $G \sim \Gamma_S(b)$ are independent.

The existence of a Lévy process satisfying this definition is ensured by Proposition 1.3.2. Now we outline some alternative characterisations of a VG process in terms of its characteristic triplet and characteristic exponent, and give a formula for its Lévy density. Let

$$\mathfrak{K}_\rho(r) := r^\rho K_\rho(r), \quad \rho \geq 0, \quad r > 0, \tag{1.3.2}$$

where K_ρ is a modified Bessel function of the second kind (see Section A.1). Recall that $\langle \mathbf{x}, \mathbf{y} \rangle_\Sigma := \mathbf{x}\Sigma\mathbf{y}'$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\Sigma \in \mathbb{R}^{n\times n}$. Recall that $\mathrm{d}\mathbf{x}$ is the Lebesgue measure on $\mathbb{R}^n_*$.

Proposition 1.3.4. *Let $b > 0$, $\boldsymbol{\mu} \in \mathbb{R}^n$ and $\Sigma \in \mathbb{R}^{n\times n}$ be a covariance matrix. The following are equivalent:*

(i) $\mathbf{V} \sim VG^n(b, \boldsymbol{\mu}, \Sigma)$;

(ii) $\mathbf{V} \sim FV^n(\mathbf{0}, \mathcal{V}_{b,\boldsymbol{\mu},\Sigma})$ *with Lévy measure*

$$\mathcal{V}_{b,\boldsymbol{\mu},\Sigma}(\mathrm{d}\mathbf{x}) = \int_{(0,\infty)} \mathbb{P}(\mathbf{B}(g) \in \mathrm{d}\mathbf{x}) b e^{-bg} \frac{\mathrm{d}g}{g}, \tag{1.3.3}$$

where $\mathbf{B} \sim BM^n(\boldsymbol{\mu}, \Sigma)$;

(iii) $\mathbf{V}$ *is an n-dimensional Lévy process with characteristic exponent*

$$\Psi_{\mathbf{V}}(\boldsymbol{\theta}) = -b \ln\left(1 - \frac{\mathrm{i}\langle \boldsymbol{\mu}, \boldsymbol{\theta} \rangle}{b} + \frac{\|\boldsymbol{\theta}\|^2_\Sigma}{2b}\right), \quad \boldsymbol{\theta} \in \mathbb{R}^n. \tag{1.3.4}$$

If (i)–(iii) are satisfied and Σ is invertible, then $\mathcal{V}_{b,\boldsymbol{\mu},\Sigma}$ is absolutely continuous with respect to $\mathrm{d}\mathbf{x}$*, having Lévy density*

$$\frac{\mathrm{d}\mathcal{V}_{b,\boldsymbol{\mu},\Sigma}}{\mathrm{d}\mathbf{x}}(\mathbf{x}) = \frac{2b\exp(\langle \boldsymbol{\mu}, \mathbf{x} \rangle_{\Sigma^{-1}})}{(2\pi)^{n/2}\|\mathbf{x}\|^n_{\Sigma^{-1}}|\Sigma|^{1/2}} \mathfrak{K}_{n/2}((2b + \|\boldsymbol{\mu}\|^2_{\Sigma^{-1}})^{1/2}\|\mathbf{x}\|_{\Sigma^{-1}}), \quad \mathbf{x} \in \mathbb{R}^n_*. \tag{1.3.5}$$

Proof. For (i) ⇔ (ii), see Theorem 30.1 and Equation (30.8) in [Sat99]. For (i) ⇔ (iii), see Equation (2.9) in [BKMS17]. For Lévy density, see Equation (2.11) in [BKMS17]. □

The following example shows that the result of strong subordination can fail to be a Lévy process.

Example 1.3.5. If B is a standard Brownian motion and I is the identity function, then $(I, 2I)$ is a subordinator and (B, B) is a Lévy process, but $\mathbf{Y} := (B, B) \circ (I, 2I)$ is not a Lévy process. For instance, $\mathbf{Y}$ does not have independent increments since $\mathrm{Cov}(Y_2(1) - Y_2(0), Y_1(2) - Y_1(1)) = 1 \neq 0$. While $\mathbf{Y}$ is a Gaussian process, it is not a Brownian motion.

We now derive necessary conditions for strong subordination to produce a Lévy process. Proposition 1.3.6 below says that under the usual assumptions of strong subordination listed in Definition 1.3.1, in addition to any of the assumptions (i)–(iii) being satisfied, in situations outside of the sufficient conditions of Proposition 1.3.2, specifically, when $\mathbf{T}$ and $\mathbf{X}$ are n-dimensional with $n \geq 2$, $\mathbf{T}$ has nonzero components that are not indistinguishable and all pairs of components of $\mathbf{X}$ are dependent, then $\mathbf{X} \circ \mathbf{T}$ cannot be a Lévy process. In this sense, strong subordination is not closed. Note that the assumptions (i)–(iii) cover a wide range of cases and Example 1.3.5 satisfies all of them.

Proposition 1.3.6. *Suppose $n \geq 2$. Let $\mathbf{T}$ and $\mathbf{X}$ be n-dimensional Lévy processes, where $\mathbf{T}$ and $\mathbf{X}$ are independent, $\mathbf{T}$ is a subordinator and all pairs of components of $\mathbf{X}$ are dependent. Assume that all components of $\mathbf{T}$ are nonzero. If $\mathbf{X} \circ \mathbf{T}$ is a Lévy process, then $\mathbf{T}$ has indistinguishable components provided that one of the following holds:*

(i) $\mathbf{X} \stackrel{D}{=} -\mathbf{X}$ *is symmetric;*

(ii) $\mathbf{T}$ *is deterministic;*

(iii) $\mathbf{T}$ *admits a finite first moment,* $\mathbf{X}$ *admits a finite second moment and all pairs of components of* $\mathbf{X}$ *are correlated.*

Proof. Let $\mathbf{T} = (T_1, \ldots, T_n)$ and $\mathbf{X} = (X_1, \ldots, X_n)$. For each hypothesis in this proposition, the corresponding hypothesis obtained by replacing $\mathbf{T}$ and $\mathbf{X}$ with (T_k, T_l) and (X_k, X_l), respectively, for any $1 \leq k \neq l \leq n$, also holds. If the corresponding hypotheses for (T_k, T_l) and (X_k, X_l) imply that $T_k = T_l$ are indistinguishable for $1 \leq k \neq l \leq n$, then $\mathbf{T}$ has indistinguishable components. Hence, we can assume without loss of generality that $n = 2$.

Let the bivariate subordinator $\mathbf{T} = (T_1, T_2)$ and the bivariate Lévy process $\mathbf{X} = (X_1, X_2)$ be independent. Let $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}^2$, $r \geq 0$, $0 \leq s \leq t$. Let

$$\begin{aligned} \widehat{\Psi}_{\mathbf{X}}(\boldsymbol{\theta}) &:= \Psi_{\mathbf{X}}(\boldsymbol{\theta}) - \Psi_{X_1}(\theta_1) - \Psi_{X_2}(\theta_2), \\ A(s,t) &:= (T_1(s) \wedge T_2(t)) - (T_1(s) \wedge T_2(s)), \\ Z(s,t,\boldsymbol{\theta}) &:= T_1(s)\Psi_{X_1}(\theta_1) + (T_2(t) - T_2(s))\Psi_{X_2}(\theta_2). \end{aligned} \tag{1.3.6}$$

Using (1.2.6), we have

$$\begin{aligned} &(r,t,s) \diamond \Psi_{X_1,X_2,X_2}(\boldsymbol{\theta}, -\theta_2) \\ &\quad = \mathbf{1}_{\{r<s\}}(r,t,s)(r\Psi_{X_1}(\theta_1) + (t-s)\Psi_{X_2}(\theta_2)) \end{aligned}$$

$$
\begin{aligned}
&+ \mathbf{1}_{\{s\le r\le t\}}(r,t,s)(s\Psi_{X_1}(\theta_1) + (r-s)\Psi_{\mathbf{X}}(\boldsymbol{\theta}) + (t-r)\Psi_{X_2}(\theta_2)) \\
&+ \mathbf{1}_{\{r>t\}}(r,t,s)(s\Psi_{X_1}(\theta_1) + (t-s)\Psi_{\mathbf{X}}(\boldsymbol{\theta}) + (r-t)\Psi_{X_1}(\theta_1)) \\
= {}& r\Psi_{X_1}(\theta_1) + (t-s)\Psi_{X_2}(\theta_2) + (r\wedge t - r\wedge s)\widehat{\Psi}_{\mathbf{X}}(\boldsymbol{\theta}),
\end{aligned}
$$

and thus, by conditioning on $\mathbf{T}$ and using Proposition 1.2.2, we have

$$
\begin{aligned}
\Phi_{(X_1\circ T_1(s), X_2\circ T_2(t)-X_2\circ T_2(s))}(\boldsymbol{\theta}) &= \mathbb{E}[\exp((T_1(s),T_2(t),T_2(s))\diamond\Psi_{X_1,X_2,X_2}(\boldsymbol{\theta},-\theta_2))] \\
&= \mathbb{E}[\exp(Z(s,t,\boldsymbol{\theta}) + \widehat{\Psi}_{\mathbf{X}}(\boldsymbol{\theta})A(s,t))].
\end{aligned}
$$

On the other hand, by noting that $X_1\circ T_1(s)$ and $X_2\circ T_2(t) - X_2\circ T_2(s)$ are independent as $\mathbf{X}\circ\mathbf{T}$ is assumed to be a Lévy process, and then conditioning on $\mathbf{T}$, the LHS above can also be written as

$$
\begin{aligned}
&\mathbb{E}[\mathbb{E}[e^{\mathrm{i}\theta_1 X_1\circ T_1(s)}|T_1(s)]\mathbb{E}[e^{\mathrm{i}\theta_2(X_2\circ T_2(t)-X_2\circ T_2(s))}|T_2(s),T_2(t)]] \\
&\quad = \mathbb{E}[\exp(T_1(s)\Psi_{X_1}(\theta_1))\exp((T_2(t)-T_2(s))\Psi_{X_2}(\theta_2))] \\
&\quad = \mathbb{E}[\exp(Z(s,t,\boldsymbol{\theta}))].
\end{aligned}
$$

The second line is obtained using the stationary increment property of X_2. To summarise,

$$
\mathbb{E}[\exp(Z(s,t,\boldsymbol{\theta}))] = \mathbb{E}[\exp(Z(s,t,\boldsymbol{\theta}) + \widehat{\Psi}_{\mathbf{X}}(\boldsymbol{\theta})A(s,t))], \tag{1.3.7}
$$

for all $\boldsymbol{\theta}\in\mathbb{R}^2$, $r\ge 0$, $0\le s\le t$.

(i). Assume $\mathbf{X} \stackrel{D}{=} -\mathbf{X}$. Since X_1 and X_2 are dependent, there exists $\boldsymbol{\theta}\in\mathbb{R}^2$ such that $\widehat{\Psi}_{\mathbf{X}}(\boldsymbol{\theta}) \ne 0$. By symmetry, $\Psi_{\mathbf{X}}(\boldsymbol{\theta}), \widehat{\Psi}_{\mathbf{X}}(\boldsymbol{\theta}), \Psi_{X_k}(\theta_k) \in \mathbb{R}$, $k = 1, 2$. Let $t > 0$, $u > 1$. In (1.3.7), we have $Z(t, ut, \boldsymbol{\theta}) \in \mathbb{R}$, forcing $A(t, ut) = 0$ a.s., which we consider in the three cases, $T_1(t) < T_2(t)$, $T_2(t) \le T_1(t) \le T_2(ut)$, $T_1(t) > T_2(ut)$. As T_2 cannot degenerate to a zero process, we must have $T_2(t) < T_2(ut)$ for some $u > 1$, and when $A(t, ut) = 0$, the case $T_1(t) > T_2(ut)$ cannot occur. This happens with probability one because $u \mapsto A(t, ut)$ degenerates to a zero process. Thus, by considering the two remaining cases, we must have $T_1(t) \le T_2(t)$ a.s. Reversing the role of T_1 and T_2 completes the proof of Part (i).

(ii). Since X_1 and X_2 are dependent, there exists $\boldsymbol{\theta}\in\mathbb{R}^2$ such that $\widehat{\Psi}_{\mathbf{X}}(\boldsymbol{\theta}) \ne 0$. If $\mathbf{T}$ is deterministic with drift (d_1, d_2), then (1.3.7) implies $\widehat{\Psi}_{\mathbf{X}}(\boldsymbol{\theta})A(t, (1+\varepsilon)t) \in 2\pi\mathrm{i}\mathbb{Z}$ for all $t, \varepsilon > 0$, which we consider in the three cases, $d_1 < d_2$, $d_2 \le d_1 \le d_2(1+\varepsilon)$, $d_1 > d_2(1+\varepsilon)$. As T_2 cannot degenerate to a zero process, it is not possible that $\widehat{\Psi}_{\mathbf{X}}(\boldsymbol{\theta})d_2 t\varepsilon \in 2\pi\mathrm{i}\mathbb{Z}$ for all $t, \varepsilon > 0$, so the case $d_1 > d_2(1+\varepsilon)$ cannot occur. Thus, by considering the two remaining cases, we must have $d_1 \le d_2$. Reversing the role of T_1

and T_2 completes the proof of Part (ii).

(iii). Let $t > 0$, $u > 1$. Suppose there exists a sequence $\boldsymbol{\theta}_m \to \mathbf{0}$ as $m \to \infty$ such that $\widehat{\Psi}_{\mathbf{X}}(\boldsymbol{\theta}_m) \neq 0$, $\Re\widehat{\Psi}_{\mathbf{X}}(\boldsymbol{\theta}_m) \leq 0$ and $\Re\Psi_{X_k}(\theta_k) \leq 0$ for all $k = 1, 2$, $m \in \mathbb{N}$. Since $|\exp(Z(t, ut, \boldsymbol{\theta}_m))| \leq 1$ and $|1 - e^z| \leq |z|$ for $\Re z \leq 0$ from Lemma A.3.4, we have

$$\left| \exp(Z(t, ut, \boldsymbol{\theta}_m)) \frac{1 - \exp(\widehat{\Psi}_{\mathbf{X}}(\boldsymbol{\theta}_m) A(t, ut))}{\widehat{\Psi}_{\mathbf{X}}(\boldsymbol{\theta}_m)} \right| \leq A(t, ut).$$

Since $\mathbf{T}(1)$ admits a finite first moment, so does $A(t, ut)$. Thus, the dominated convergence theorem is applicable, giving

$$\lim_{m \to \infty} \mathbb{E}\left[\exp(Z(t, ut, \boldsymbol{\theta}_m)) \frac{1 - \exp(\widehat{\Psi}_{\mathbf{X}}(\boldsymbol{\theta}_m) A(t, ut))}{\widehat{\Psi}_{\mathbf{X}}(\boldsymbol{\theta}_m)} \right] = \mathbb{E}[A(t, ut)].$$

The LHS is 0 by (1.3.7), so $\mathbb{E}[A(t, ut)] = 0$. Thus, noting that $A(t, ut) \geq 0$ a.s. gives $A(t, ut) = 0$ a.s.

By assumption, $\mathbf{X}(1)$ has a finite second moment. So using a Taylor series expansion, we have $\Re\widehat{\Psi}_{\mathbf{X}}(\boldsymbol{\theta}) = -\rho\theta_1\theta_2 + o(\|\boldsymbol{\theta}\|^2)$ and $\Re\Psi_{X_k}(\theta_k) = -\sigma_k^2\theta_k^2/2 + o(\theta_k^2)$, $k = 1, 2$, as $\boldsymbol{\theta} \to \mathbf{0}$, where $\rho := \mathrm{Cov}(X_1(1), X_2(1)) \neq 0$ and $\sigma_k^2 := \mathrm{Var}(X_k(1)) \neq 0$ by the assumption that X_1 and X_2 are correlated. Thus, it is always possible to construct a sequence $\boldsymbol{\theta}_m \to \mathbf{0}$ satisfying the requirements in the previous paragraph.

To summarise, we have $A(t, ut) = 0$ a.s. for all $t > 0$, $u > 1$. Then the proof is completed as in Part (i). □

Remark 1.3.7. Let B be standard Brownian motion and I be the identity function. Then $(B, B) \circ (I, 0)$ is a Lévy process. This demonstrates the necessity of assuming that all components of the subordinator are nonzero in Proposition 1.3.6.

Remark 1.3.8. Let B, B^* be independent standard Brownian motions and I be the identity function. Note that (B, B, B^*) has dependent components and $(I, I, 2I)$ does not have indistinguishable components, but $(B, B, B^*) \circ (I, I, 2I)$ is a Lévy process. Thus, the assumption that all pairs of components of $\mathbf{X}$ are dependent in Proposition 1.3.6 cannot be replaced with the assumption that $\mathbf{X}$ has dependent components. However, the assumption can be weakened to having sufficiently many pairs (X_k, X_l) of dependent components of $\mathbf{X}$ such that $T_k = T_l$ for all these pairs implies that $\mathbf{T}$ has indistinguishable components.

Chapter 2

Weak Subordination

The strong subordination of $\mathbf{X}$ and $\mathbf{T}$ produces a Lévy process $\mathbf{X} \circ \mathbf{T}$ when the subordinate $\mathbf{X}$ has independent components or the subordinator $\mathbf{T}$ has indistinguishable components. This chapter introduces the weak subordination of $\mathbf{X}$ and $\mathbf{T}$, which extends this notion in a way that always produces a Lévy process $\mathbf{X} \odot \mathbf{T}$ and matches strong subordination in law in the previous cases. For increased generality, we work with the joint process $(\mathbf{T}, \mathbf{X} \odot \mathbf{T})$.

In Section 2.1, we outline a heuristic construction of weak subordination using marked Poisson point processes. In Section 2.2, a rigorous proof of the existence of the Lévy process $\mathbf{X} \odot \mathbf{T}$ is given. In Section 2.3, we develop some useful properties of weak subordination, including its characteristics and its consistency with strong subordination, among others. The chapter ends with a discussion of the case where the subordinator has monotonic components. Here, the weakly subordinated process has the property that its distribution matches that of the corresponding strongly subordinated process at all time points. We show that this property does not hold in general, and in some cases no Lévy process has this property.

2.1 Construction

We give a brief, heuristic construction of the weak subordination of $\mathbf{X} \sim L^n(\boldsymbol{\mu}, \Sigma, \mathcal{X})$ and $\mathbf{T} \sim S^n(\mathbf{d}, \mathcal{T})$ based on the idea of assigning the law of $\mathbf{X}(\mathbf{t})$ to the weakly subordinated process conditional on the subordinator taking the value $\mathbf{T}(t) = \mathbf{t}$ at time $t \geq 0$. Denote the weakly subordinated process by $\mathbf{X} \odot \mathbf{T}$. Recall that the jump process $\Delta\mathbf{X}$ is defined by $\Delta\mathbf{X}(t) := \mathbf{X}(t) - \mathbf{X}(t-)$, $t > 0$, where $\mathbf{X}(t-) := \lim_{s \uparrow t} \mathbf{X}(s)$.

It is always possible to write $\mathbf{T} = I\mathbf{d} + \mathbf{S}$, where $I\mathbf{d}$ is a deterministic subordinator and $\mathbf{S} \sim S^n(\mathbf{0}, \mathcal{T})$ is a pure-jump subordinator.

Suppose $\mathbf{S} \equiv \mathbf{0}$ and $\mathbf{T} = I\mathbf{d}$. A Lévy process $\mathbf{Y}$ that has the same distribution as

$\mathbf{X}(\mathbf{t})$, when $\mathbf{T}(t) = \mathbf{t}$, satisfies $\mathbf{Y}(t) \stackrel{D}{=} \mathbf{X}(t\mathbf{d})$ for all $t \geq 0$. For this to hold, we must have $\mathbf{Y} \sim L^n(\mathbf{d} \diamond \boldsymbol{\mu} + \mathbf{c}, \mathbf{d} \diamond \Sigma, \mathbf{d} \diamond \mathcal{X})$ due to Proposition 1.2.2. We take $\mathbf{X} \odot \mathbf{T} \stackrel{D}{=} \mathbf{Y}$, which determines the law of the weakly subordinated Lévy process in the case of deterministic subordinators.

Now suppose $\mathbf{d} = \mathbf{0}$ and $\mathbf{T} = \mathbf{S}$. Let $\mathbf{Z} = (\mathbf{S}, \mathbf{Y})$ be a $2n$-dimensional Lévy process on $[0, \infty)^n \times \mathbb{R}^n$. Under strong subordination, the jumps of $\mathbf{Y}$ are determined pathwisely by

$$\Delta \mathbf{Y}(t) = \mathbf{X}(\mathbf{T}(t)) - \mathbf{X}(\mathbf{T}(t-)), \quad t > 0,$$

and may not produce a Lévy process. Informally, under weak subordination, we equate the law of the LHS and RHS, conditional on $\Delta \mathbf{T}(t) = \mathbf{t}$, so that the jumps of $\mathbf{Y}$ have conditional law

$$\mathbb{P}(\Delta \mathbf{Y}(t) \in \mathrm{d}\mathbf{x} \,|\, \Delta \mathbf{T}(t) = \mathbf{t}) = \mathbb{P}(\mathbf{X}(\mathbf{t}) \in \mathrm{d}\mathbf{x}), \quad t > 0.$$

Formally, it will be shown that this turns out to mean that $\mathbf{Y}$ has Lévy measure

$$\mathcal{Y}(\mathrm{d}\mathbf{x}) = \int_{[0,\infty)^n_*} \mathbb{P}(\mathbf{X}(\mathbf{t}) \in \mathrm{d}\mathbf{x}) \mathcal{T}(\mathrm{d}\mathbf{t}),$$

where $\mathcal{T}$ is the Lévy measure of $\mathbf{T}$ (see Proposition 2.3.4). This is a property that strong subordination also satisfies as seen in (1.3.1).

Now we describe the construction of a Lévy process having this property. The jumps of the subordinator $(t, \Delta \mathbf{T}(t))_{t>0, \Delta \mathbf{T}(t) \neq \mathbf{0}}$ forms a Poisson point process on $[0, \infty) \times [0, \infty)^n_*$ with intensity measure $\mathrm{d}t \otimes \mathcal{T}$. Now the jumps of the joint process $(t, \Delta \mathbf{Z}(t))_{t>0, \Delta \mathbf{Z}(t) \neq \mathbf{0}}$ can be formed as the marked Poisson point process with marks on $\mathbb{R}^n$ having law $\mathbb{P}(\mathbf{X}(\Delta \mathbf{T}(t)) \in \mathrm{d}\mathbf{x})$ when $t > 0$ and $\Delta \mathbf{T}(t) \neq \mathbf{0}$ that are conditionally independent given the jumps of the subordinator $(t, \Delta \mathbf{T}(t))_{t>0, \Delta \mathbf{T}(t) \neq \mathbf{0}}$. In this situation, the marked Poisson point process $(t, \Delta \mathbf{Z}(t))_{t>0, \Delta \mathbf{Z}(t) \neq \mathbf{0}}$ can be associated to a pure-jump Lévy process $\mathbf{Z}$ taking values on $[0, \infty)^n \times \mathbb{R}^n$ through its Lévy-Itô decomposition, which sums up those jumps for $t > 0$, possibly with a compensation term. We take $(\mathbf{T}, \mathbf{X} \odot \mathbf{T}) \stackrel{D}{=} \mathbf{Z}$, in particular $\mathbf{X} \odot \mathbf{T} \stackrel{D}{=} \mathbf{Y}$, which determines the law of the weakly subordinated Lévy process in the case of pure-jump subordinators.

For a general subordinator $\mathbf{T} = I\mathbf{d} + \mathbf{S}$, we take the law of the weakly subordinated process to be the convolution of the laws of the weakly subordinated process for the deterministic subordinator $I\mathbf{d}$ and the pure-jump subordinator $\mathbf{S}$ as determined above. This is a property also enjoyed by univariate and multivariate subordination (see Proposition 4.3 in [BKMS17]).

In general, it is not possible to simply define weak subordination as the Lévy process $\mathbf{Y}$ satisfying $\mathbf{Y}(t) \stackrel{D}{=} \mathbf{X} \circ \mathbf{T}(t)$ for all $t \geq 0$ as we will see in Example 2.3.28 and Proposition 2.3.29.

2.2 Existence

The discussion in Section 2.1 motivates the definition for weak subordination in terms of a characteristic triplet. The proof of Theorem 2.2.4 (ii) and Remark 2.2.5 below will show that this definition is consistent with the construction outlined above.

Let $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2) \sim L^{2n}(\mathbf{m}, \Sigma, \mathcal{Z})$ be a Lévy process on $\mathbb{R}^{2n}$ for some $\mathbf{m} \in \mathbb{R}^{2n}$, covariance matrix $\Sigma \in \mathbb{R}^{2n \times 2n}$ and Lévy measure $\mathcal{Z}$ on $\mathbb{R}^{2n}_*$. The projected n-dimensional processes $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are Lévy processes. Our notation extends from $\mathbb{R}^n$ to $\mathbb{R}^{2n}$ in the usual way. In particular, $\|\cdot\|$ and $\mathbb{D}$ may denote the Euclidean norm and the Euclidean unit ball in $\mathbb{R}^n$ or $\mathbb{R}^{2n}$, respectively.

Definition 2.2.1. Let $\mathbf{X} \sim L^n(\boldsymbol{\mu}, \Sigma, \mathcal{X})$ and $\mathbf{T} \sim S^n(\mathbf{d}, \mathcal{T})$. A process

$$\mathbf{Z} \stackrel{D}{=} (\mathbf{T}, \mathbf{X} \odot \mathbf{T})$$

is the *weak subordination* of $\mathbf{X}$ and $\mathbf{T}$ if $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2) \sim L^{2n}(\mathbf{m}, \Theta, \mathcal{Z})$ is a Lévy process with characteristic triplet determined by

$$\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2), \tag{2.2.1}$$

$$\mathbf{m}_1 = \mathbf{d} + \int_{[0,\infty)^n_*} \mathbf{t}\mathbb{P}((\mathbf{t}, \mathbf{X}(\mathbf{t})) \in \mathbb{D})\, \mathcal{T}(\mathrm{d}\mathbf{t}), \tag{2.2.2}$$

$$\mathbf{m}_2 = \mathbf{c}(\mathbf{d}, \mathcal{X}) + \mathbf{d} \diamond \boldsymbol{\mu} + \int_{[0,\infty)^n_*} \mathbb{E}[\mathbf{X}(\mathbf{t}) \mathbf{1}_{\mathbb{D}}(\mathbf{t}, \mathbf{X}(\mathbf{t}))]\, \mathcal{T}(\mathrm{d}\mathbf{t}), \tag{2.2.3}$$

$$\Theta = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{d} \diamond \Sigma \end{pmatrix}, \tag{2.2.4}$$

$$\mathcal{Z}(\mathrm{d}\mathbf{t}, \mathrm{d}\mathbf{x}) = (\boldsymbol{\delta}_{\mathbf{0}} \otimes (\mathbf{d} \diamond \mathcal{X}))(\mathrm{d}\mathbf{t}, \mathrm{d}\mathbf{x}) + \mathbf{1}_{[0,\infty)^n_* \times \mathbb{R}^n}(\mathbf{t}, \mathbf{x}) \mathbb{P}(\mathbf{X}(\mathbf{t}) \in \mathrm{d}\mathbf{x}) \mathcal{T}(\mathrm{d}\mathbf{t}). \tag{2.2.5}$$

The process $\mathbf{X}$ is the *subordinate*. If $\mathbf{Z}_1 = \mathbf{T}$ are indistinguishable and $\mathbf{Z} \stackrel{D}{=} (\mathbf{T}, \mathbf{X} \odot \mathbf{T})$, then $\mathbf{Z}$ is the *semi-strong subordination* of $\mathbf{X}$ and $\mathbf{T}$.

It will be shown in Theorem 2.2.4 that there exists a Lévy process $\mathbf{Z}$ determined by the characteristics in (2.2.1)–(2.2.5). Before we can prove this, Lemma 2.2.2 below collects some inequalities analogous to Lemma 30.3 in [Sat99], but adapted to deal with the multivariate time parameter. These inequalities will be used in a similar way as in Theorem 3.3 of [BNPS01], to show that $(\mathbf{m}, \Theta, \mathcal{Z})$ is a valid characteristic triplet that specifies a Lévy process.

Lemma 2.2.2. *If* $\mathbf{X} \sim L^n(\boldsymbol{\mu}, \Sigma, \mathcal{X})$ *and* $\boldsymbol{\theta} \in \mathbb{R}^n$, *then there exist finite constants* $C_1 = C_1(\boldsymbol{\theta}, \mathbf{X})$, $C_2 = C_2(\mathbf{X})$ *and* $C_3 = C_3(\mathbf{X})$ *such that, for all* $\mathbf{t} \in [0, \infty)^n$,

$$|\Phi_{\mathbf{X}(\mathbf{t})}(\boldsymbol{\theta}) - 1| \le C_1(1 \wedge \|\mathbf{t}\|), \tag{2.2.6}$$

$$\mathbb{E}[1 \wedge \|\mathbf{X}(\mathbf{t})\|^2] \le C_2(1 \wedge \|\mathbf{t}\|), \tag{2.2.7}$$

$$\mathbb{E}[1 \wedge \|\mathbf{X}(\mathbf{t})\|] \le C_2^{1/2}(1 \wedge \|\mathbf{t}\|^{1/2}), \tag{2.2.8}$$

$$\|\mathbb{E}[\mathbf{X}(\mathbf{t})\mathbf{1}_{\mathbb{D}}(\mathbf{X}(\mathbf{t}))]\| \le C_3(1 \wedge \|\mathbf{t}\|). \tag{2.2.9}$$

Proof. Let $\mathbf{t} = (t_1, \dots, t_n) \in [0, \infty)^n$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$.

Proof of (2.2.6). Introduce the Lévy measure $\mathcal{N} := \sum_{\langle(1),\dots,(n)\rangle} \sum_{k=1}^n \mathcal{X}_{\{(k),\dots,(n)\}}$ with the outer summation taken over all permutations $\langle(1), \dots, (n)\rangle$.

Let $z := \mathbf{t} \diamond \Psi_{\mathbf{X}}(\boldsymbol{\theta})$ in (1.2.7). Note that $\mathbf{t} \diamond \Sigma$ is a covariance matrix by Lemma 1.2.1, so that $\|\boldsymbol{\theta}\|^2_{\mathbf{t}\diamond\Sigma} \ge 0$. Now since

$$\Re(\mathbf{t} \diamond \Psi(\boldsymbol{\theta})) = -\frac{1}{2}\|\boldsymbol{\theta}\|^2_{\mathbf{t}\diamond\Sigma} - \int_{\mathbb{R}^n_*} (1 - \cos\langle\boldsymbol{\theta}, \mathbf{x}\rangle)\, (\mathbf{t} \diamond \mathcal{X})(\mathrm{d}\mathbf{x}) \le 0,$$

Lemma A.3.4 can be applied, giving $|e^z - 1| \le |z|$. Further, we have $|\Re(\mathbf{t} \diamond \Psi(\boldsymbol{\theta}))| \le C_{11}\|\mathbf{t}\|$, where

$$C_{11} := \frac{1}{2}\sum_{k=1}^n \sum_{l=1}^n |\theta_k \theta_l \Sigma_{kl}| + \int_{\mathbb{R}^n_*} |1 - \cos\langle\boldsymbol{\theta}, \mathbf{x}\rangle|\, \mathcal{N}(\mathrm{d}\mathbf{x}).$$

Recalling (1.2.4), we have $|\Im(\mathbf{t} \diamond \Psi(\boldsymbol{\theta}))| \le C_{12}\|\mathbf{t}\|$, where

$$C_{12} := n(\|\boldsymbol{\mu}\| + n^{1/2}\mathcal{X}(\mathbb{D}^C))\|\boldsymbol{\theta}\| + \int_{\mathbb{R}^n_*} |\langle\boldsymbol{\theta}, \mathbf{x}\rangle \mathbf{1}_{\mathbb{D}}(\mathbf{x}) - \sin\langle\boldsymbol{\theta}, \mathbf{x}\rangle|\, \mathcal{N}(\mathrm{d}\mathbf{x}).$$

Since $\mathcal{N}$ is a Lévy measure, the integrand in the Lévy-Khintchine formula (1.1.2) with $\mathcal{X}$ replaced by $\mathcal{N}$, is integrable, which implies that its real and imaginary part are also integrable, so the integrals in C_{11} and C_{12} are finite (see Remark 8.4 in [Sat99]). Recalling that $\mathcal{X}(\mathbb{D}^C)$ is also finite by (1.1.3), C_{11} and C_{12} are finite constants. Choosing $C_{13} := (C_{11}^2 + C_{12}^2)^{1/2}$ shows that $|\Phi_{\mathbf{X}(\mathbf{t})}(\boldsymbol{\theta}) - 1| \le C_{13}\|\mathbf{t}\|$. Since characteristic functions are bounded, we also have $|\Phi_{\mathbf{X}(\mathbf{t})}(\boldsymbol{\theta}) - 1| \le 2$. Thus, (2.2.6) holds with $C_1 := C_{13} + 2$.

Proof of (2.2.7). Define the Lévy measures $\mathcal{Y}_{\mathbf{t}}$ and $\mathcal{Z}_{\mathbf{t}}$ as the restriction of $\mathbf{t} \diamond \mathcal{X}$ to $\mathbb{D}^C$ and $\mathbb{D}$, respectively, that is $\mathcal{Y}_{\mathbf{t}}(A) := (\mathbf{t} \diamond \mathcal{X})(A \cap \mathbb{D}^C)$ and $\mathcal{Z}_{\mathbf{t}}(A) := (\mathbf{t} \diamond \mathcal{X})(A \cap \mathbb{D})$

for Borel sets $A \subseteq \mathbb{R}^n_*$. Recalling $\mathbf{c} = (c_1, \dots, c_n) = \mathbf{c}(\mathbf{t}, \mathcal{X})$, let $\mathbf{Y}^{(\mathbf{t})} \sim L^n(\mathbf{0}, 0, \mathcal{Y}_\mathbf{t})$ and $\mathbf{Z}^{(\mathbf{t})} = (Z_1^{(\mathbf{t})}, \dots, Z_n^{(\mathbf{t})}) \sim L^n(\mathbf{t} \diamond \boldsymbol{\mu} + \mathbf{c}, \mathbf{t} \diamond \Sigma, \mathcal{Z}_\mathbf{t})$. By Proposition 1.2.2, we may decompose $\mathbf{X}(\mathbf{t}) \overset{D}{=} \mathbf{Y}^{(\mathbf{t})}(1) + \mathbf{Z}^{(\mathbf{t})}(1)$ into a sum of independent n-dimensional random vectors.

Note that $\mathbf{Y}^{(\mathbf{t})}$ is a compound Poisson process with jumps in norm larger than 1, and it is determined by a rate parameter λ and a jump size distribution $\mathcal{P}$ satisfying $\mathcal{Y}_\mathbf{t} = \lambda \mathcal{P}$, which implies $\lambda = (\mathbf{t} \diamond \mathcal{X})(\mathbb{D}^C)$. Therefore,

$$\mathbb{P}(\mathbf{Y}^{(\mathbf{t})}(1) = \mathbf{0}) \geq \mathbb{P}\{\mathbf{Y}^{(\mathbf{t})} \text{ has no jumps in the time interval } [0,1]\} = e^{-\lambda}.$$

Since $(\mathbf{t} \diamond \mathcal{X})(\mathbb{D}^C) \leq \mathcal{N}(\mathbb{D}^C)\|\mathbf{t}\|$ and $1 - e^{-x} \leq x$, $x \in \mathbb{R}$, we have

$$\mathbb{P}(\mathbf{Y}^{(\mathbf{t})}(1) \neq \mathbf{0}) \leq 1 - e^{\lambda} \leq \mathcal{N}(\mathbb{D}^C)\|\mathbf{t}\|. \tag{2.2.10}$$

On the other hand, $\mathbf{Z}^{(\mathbf{t})}$ has jumps bounded in norm by 1. In particular, $\mathbf{Z}^{(\mathbf{t})}(1)$ has finite moments of all order (see Corollary 25.8 in [Sat99]). By Proposition 1.1.9, we have

$$\mathbb{E}[Z_k^{(\mathbf{t})}(1)] = \mu_k t_k + c_k, \quad \operatorname{Var}(Z_k^{(\mathbf{t})}(1)) = \Sigma_{kk} t_k + \int_{\mathbb{D}_*} x_k^2 \, (\mathbf{t} \diamond \mathcal{X})(\mathrm{d}\mathbf{x}), \quad 1 \leq k \leq n,$$

Using $(x+y)^2 \leq 2(x^2 + y^2)$, $x, y \in \mathbb{R}$, and then (1.2.4), we have

$$\sum_{k=1}^n (\mathbb{E}[Z_k^{(\mathbf{t})}(1)])^2 \leq C_{21}\|\mathbf{t}\|^2, \quad \sum_{k=1}^n \operatorname{Var}(Z_k^{(\mathbf{t})}(1)) \leq C_{22}\|\mathbf{t}\|,$$

where

$$C_{21} := 2\|\boldsymbol{\mu}\|^2 + 2n\mathcal{X}(\mathbb{D}^C)^2, \quad C_{22} := \operatorname{trace}(\Sigma) + \int_{\mathbb{D}_*} \|\mathbf{x}\|^2 \mathcal{N}(\mathrm{d}\mathbf{x}),$$

are finite constants, the latter due to $\mathcal{N}$ being a Lévy measure.

Combining these last two inequalities and letting $C_{23} := C_{21} + C_{22}$ yields

$$\mathbb{E}[\|\mathbf{Z}^{(\mathbf{t})}(1)\|^2] \leq C_{23}(\|\mathbf{t}\| + \|\mathbf{t}\|^2). \tag{2.2.11}$$

Using (2.2.10) and (2.2.11) and letting $C_{24} := \mathcal{N}(\mathbb{D}^C) + C_{23}$, we obtain

$$\begin{aligned}
\mathbb{E}[1 \wedge \|\mathbf{X}(\mathbf{t})\|^2] &\leq \mathbb{E}[(1 \wedge \|\mathbf{X}(\mathbf{t})\|^2)\mathbf{1}_{\mathbb{R}^n_*}(\mathbf{Y}^{(\mathbf{t})}(1))] + \mathbb{E}[(1 \wedge \|\mathbf{Z}^{(\mathbf{t})}(1)\|^2)\mathbf{1}_{\{\mathbf{0}\}}(\mathbf{Y}^{(\mathbf{t})}(1))] \\
&\leq \mathbb{P}(\mathbf{Y}^{(\mathbf{t})}(1) \neq \mathbf{0}) + \mathbb{E}[\|\mathbf{Z}^{(\mathbf{t})}(1)\|^2] \\
&\leq C_{24}(\|\mathbf{t}\| + \|\mathbf{t}\|^2).
\end{aligned}$$

Since $\mathbb{E}[1 \wedge \|\mathbf{X}(\mathbf{t})\|^2] \le 1$ and $1 \wedge (x + x^2) \le 2(1 \wedge x)$, $x \ge 0$, (2.2.7) follows with $C_2 := 2(C_{24} + 1)$.

Proof of (2.2.8). By the Cauchy-Schwarz inequality, $\mathbb{E}[1 \wedge \|\mathbf{X}(\mathbf{t})\|] \le (\mathbb{E}[1 \wedge \|\mathbf{X}(\mathbf{t})\|^2])^{1/2}$. Then applying (2.2.7) yields (2.2.8).

Proof of (2.2.9). Set $\mathbb{D}_\infty := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_\infty \le 1\}$. If $g(x) := e^{ix} - 1$, $x \in \mathbb{R}$, we have

$$\begin{aligned}\|\mathbb{E}[\mathbf{X}(\mathbf{t})\mathbf{1}_{\mathbb{D}_\infty}(\mathbf{X}(\mathbf{t}))]\|_\infty &= \max_{1\le k\le n} |\mathbb{E}[-\mathrm{i}X_k(t_k)\mathbf{1}_{\mathbb{D}_\infty}(\mathbf{X}(\mathbf{t}))]| \\ \le \max_{1\le k\le n} |\mathbb{E}[g(X_k(t_k))\mathbf{1}_{\mathbb{D}_\infty^C}(\mathbf{X}(\mathbf{t}))]| &+ \max_{1\le k\le n} |\mathbb{E}[(g(X_k(t_k)) - \mathrm{i}X_k(t_k))\mathbf{1}_{\mathbb{D}_\infty}(\mathbf{X}(\mathbf{t}))]| \\ &+ \max_{1\le k\le n} |\mathbb{E}[g(X_k(t_k))]|.\end{aligned}$$

Now we bound each of the three terms. By noting that $|g(x)| \le 2$, $x \in \mathbb{R}$, and $\mathbf{1}_{\mathbb{D}_\infty^C} \le 1 \wedge \|\cdot\|^2$, we get

$$|\mathbb{E}[g(X_k(t_k))\mathbf{1}_{\mathbb{D}_\infty^C}(\mathbf{X}(\mathbf{t}))]| \le 2\mathbb{E}[1 \wedge \|\mathbf{X}(\mathbf{t})\|^2], \quad 1 \le k \le n,$$

and then (2.2.7) can be applied. Next, by noting that $|g(x) - \mathrm{i}x|^2 \le x^2/2$, $0 \le x \le 1$ (see Equation (8.9) in [Sat99]), we get

$$|\mathbb{E}[(g(X_k(t_k)) - \mathrm{i}X_k(t_k))\mathbf{1}_{\mathbb{D}_\infty}(\mathbf{X}(\mathbf{t}))]| \le \mathbb{E}[1 \wedge X_k^2(t_k)] \le \mathbb{E}[1 \wedge \|\mathbf{X}(\mathbf{t})\|^2], \quad 1 \le k \le n,$$

and then (2.2.7) can be applied. Lastly, we have $|\mathbb{E}[g(X_k(t_k))]| = |\Phi_{\mathbf{X}(\mathbf{t})}(\mathbf{e}_k) - 1|, 1 \le k \le n$, and then (2.2.6) can be applied with $\boldsymbol{\theta} \in \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Combining the above yields

$$\|\mathbb{E}[\mathbf{X}(\mathbf{t})\mathbf{1}_{\mathbb{D}_\infty}(\mathbf{X}(\mathbf{t}))]\|_\infty \le C_{31}(1 \wedge \|\mathbf{t}\|) \tag{2.2.12}$$

for some finite constant C_{31}.

By using (1.2.5), we get

$$\|\mathbb{E}[\mathbf{X}(\mathbf{t})\mathbf{1}_{\mathbb{D}_\infty\setminus\mathbb{D}}(\mathbf{X}(\mathbf{t}))]\| \le n^{1/2}\mathbb{E}[\|\mathbf{X}(\mathbf{t})\|_\infty \mathbf{1}_{\mathbb{D}_\infty\setminus\mathbb{D}}(\mathbf{X}(\mathbf{t}))] \le n^{1/2}\mathbb{E}[\mathbf{1}_{\mathbb{D}^C}(\mathbf{X}(\mathbf{t}))].$$

Then noting that $\mathbf{1}_{\mathbb{D}^C} \le 1 \wedge \|\cdot\|^2$ and using (2.2.7), we have

$$\|\mathbb{E}[\mathbf{X}(\mathbf{t})\mathbf{1}_{\mathbb{D}_\infty\setminus\mathbb{D}}(\mathbf{X}(\mathbf{t}))]\| \le n^{1/2}C_2(1 \wedge \|\mathbf{t}\|). \tag{2.2.13}$$

Finally, by the Euclidean triangle inequality,

$$\|\mathbb{E}[\mathbf{X}(\mathbf{t})\mathbf{1}_{\mathbb{D}}(\mathbf{X}(\mathbf{t}))]\| \le n^{1/2}\|\mathbb{E}[\mathbf{X}(\mathbf{t})\mathbf{1}_{\mathbb{D}_\infty}(\mathbf{X}(\mathbf{t}))]\|_\infty + \|\mathbb{E}[\mathbf{X}(\mathbf{t})\mathbf{1}_{\mathbb{D}_\infty\setminus\mathbb{D}}(\mathbf{X}(\mathbf{t}))]\|,$$

so combining (2.2.12) and (2.2.13) yields (2.2.9) with the finite constant $C_3 := n^{1/2}(C_{31} + C_2)$. This completes the proof of the lemma. □

Lemma 2.2.3. *Let* $\mathbf{X} \sim L^n(\boldsymbol{\mu}, \Sigma, \mathcal{X})$ *and* $\mathbf{T} \sim S^n(\mathbf{d}, \mathcal{T})$*. Then*

$$\mathcal{Z}_0(\mathrm{d}\mathbf{t}, \mathrm{d}\mathbf{x}) := \mathbf{1}_{[0,\infty)^n_* \times \mathbb{R}^n}(\mathbf{t}, \mathbf{x})\mathbb{P}(\mathbf{X}(\mathbf{t}) \in \mathrm{d}\mathbf{x})\mathcal{T}(\mathrm{d}\mathbf{t}) \tag{2.2.14}$$

defines a measure on $[0,\infty)^n \times \mathbb{R}^n$ *with*

$$\int_{([0,\infty)^n\times\mathbb{R}^n)_*} 1 \wedge \|(\mathbf{t}, \mathbf{x})\|^2 \, \mathcal{Z}_0(\mathrm{d}\mathbf{t}, \mathrm{d}\mathbf{x}) = \int_{[0,\infty)^n_*} \mathbb{E}[1 \wedge \|(\mathbf{t}, \mathbf{X}(\mathbf{t}))\|^2] \, \mathcal{T}(\mathrm{d}\mathbf{t}). \tag{2.2.15}$$

Proof. Let $\boldsymbol{\theta} \in \mathbb{R}^n$. From Proposition 1.2.2, the function $\mathbf{t} \mapsto (\mathbf{t} \diamond \Psi_{\mathbf{X}})(\boldsymbol{\theta})$ is continuous with domain $[0,\infty)^n$, so for all sequences $\mathbf{t}_m \to \mathbf{t}$ as $m \to \infty$ with $\mathbf{t}_m, \mathbf{t} \in [0,\infty)^n$, $m \in \mathbb{N}$, we have $\Phi_{\mathbf{X}(\mathbf{t}_m)}(\boldsymbol{\theta}) \to \Phi_{\mathbf{X}(\mathbf{t})}(\boldsymbol{\theta})$ as $m \to \infty$. By the Lévy continuity theorem, $\mathbf{X}(\mathbf{t}_m) \xrightarrow{D} \mathbf{X}(\mathbf{t})$ as $m \to \infty$.

Let $\mathcal{C}$ be the family of closed sets on $\mathbb{R}^n$. For all $A \in \mathcal{C}$, by the portmanteau lemma (see Lemma 2.2 in [vdV98]), $\limsup_{m\to\infty} \mathbb{P}(\mathbf{X}(\mathbf{t}_m) \in A) \le \mathbb{P}(\mathbf{X}(\mathbf{t}) \in A)$, which implies that $\mathbf{t} \mapsto \mathbb{P}(\mathbf{X}(\mathbf{t}) \in A)$ is an upper semi-continuous function for all $\mathbf{t} \in [0,\infty)^n$, and hence Borel measurable. In addition, $\sigma(\mathcal{C})$, the σ-field generated by $\mathcal{C}$, is the family of Borel sets on $\mathbb{R}^n$, and $\mathcal{C}$ is closed under intersections. Under these conditions, Lemma 1.37 in [Kal97] implies that $\mathbb{P}(\mathbf{X}(\mathbf{t}) \in \mathrm{d}\mathbf{x})$ is a Markov kernel from $[0,\infty)^n$ to $\mathbb{R}^n$, and it is also σ-finite (see page 40 in [Çın11]).

We can now apply Chapter I, Theorem 6.11 in [Çın11] to conclude that $\mathcal{Z}_0(\mathrm{d}\mathbf{t}, \mathrm{d}\mathbf{x})$ defines a measure on $[0,\infty)^n \times \mathbb{R}^n$ satisfying the Fubini-type formula

$$\int_{([0,\infty)^n\times\mathbb{R}^n)_*} 1 \wedge \|(\mathbf{t}, \mathbf{x})\|^2 \, \mathcal{Z}_0(\mathrm{d}\mathbf{t}, \mathrm{d}\mathbf{x}) = \int_{[0,\infty)^n_*} \int_{\mathbb{R}^n} 1 \wedge \|(\mathbf{t}, \mathbf{x})\|^2 \, \mathbb{P}(\mathbf{X}(\mathbf{t}) \in \mathrm{d}\mathbf{x})\mathcal{T}(\mathrm{d}\mathbf{t}),$$

and then (2.2.15) follows. □

The next theorem shows the existence of weak subordination. The main difficulty is to show that $\mathcal{Z}$ is a Lévy measure. In addition, semi-strong subordination is then always possible on an augmented probability space, and it relies on marking the Poisson point process associated to the jumps of $\mathbf{T}$.

Theorem 2.2.4. *Let* $\mathbf{X} \sim L^n(\boldsymbol{\mu}, \Sigma, \mathcal{X})$ *and* $\mathbf{T} \sim S^n(\mathbf{d}, \mathcal{T})$*.*

(i) There exists a Lévy process $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2) \sim L^{2n}(\mathbf{m}, \Theta, \mathcal{Z})$ *with* $(\mathbf{m}, \Theta, \mathcal{Z})$ *as specified in* (2.2.1)–(2.2.5)*.*

(ii) On an augmentation of the probability space on which $\mathbf{T}$ *is defined, there exists an* n*-dimensional Lévy process* $\mathbf{Z}_2$ *such that* $(\mathbf{T}, \mathbf{Z}_2)$ *is the semi-strong subordination of* $\mathbf{X}$ *and* $\mathbf{T}$*.*

Proof. (i). Since $\mathbf{d} \diamond \Sigma$ is a covariance matrix by Lemma 1.2.1, so is Θ. Next, we show that $\mathcal{Z}$ is a Lévy measure. For $\mathbf{t} \in [0,\infty)^n$, by noting $\|(\mathbf{t},\mathbf{t})\|^2 = 2\|\mathbf{t}\|^2$ and $1 \wedge \|(\mathbf{t},\mathbf{t})\| \leq 2^{1/2}(1 \wedge \|\mathbf{t}\|)$, and applying (2.2.7) to the $2n$-dimensional Lévy process $(I\mathbf{e}, \mathbf{X})$ with $C_2 := C_2((I\mathbf{e}, \mathbf{X}))$, we get

$$\mathbb{E}[1 \wedge \|(I\mathbf{e}, \mathbf{X})(\mathbf{t},\mathbf{t})\|^2] \leq C_2\,(1 \wedge \|(\mathbf{t},\mathbf{t})\|) \leq 2^{1/2} C_2\,(1 \wedge \|\mathbf{t}\|).$$

As (1.1.4) holds for $\mathcal{T}$, the RHS is $\mathcal{T}$-integrable. Hence, $\mathcal{Z}_0$ defined in (2.2.14) is a Lévy measure by Lemma 2.2.3 as (2.2.15) is finite.

Let $\mathcal{Z}_1 := \boldsymbol{\delta}_\mathbf{0} \otimes (\mathbf{d} \diamond \mathcal{X})$. Since $\mathbf{d} \diamond \mathcal{X}$ is a Lévy measure by Lemma 1.2.1, it is σ-finite. Now Fubini's theorem can be applied to $\mathcal{Z}_1$ giving

$$\int_{([0,\infty)^n \times \mathbb{R}^n)_*} 1 \wedge \|(\mathbf{t},\mathbf{x})\|^2\, \mathcal{Z}_1(\mathrm{d}\mathbf{t}, \mathrm{d}\mathbf{x}) = \int_{\mathbb{R}^n_*} 1 \wedge \|\mathbf{x}\|^2\, (\mathbf{d} \diamond \mathcal{X})(\mathrm{d}\mathbf{x}),$$

which is finite by Lemma 1.2.1. Thus, $\mathcal{Z} = \mathcal{Z}_0 + \mathcal{Z}_1$ is a Lévy measure.

Note that

$$\|\mathbf{t}\|\, \mathbb{P}((\mathbf{t}, \mathbf{X}(\mathbf{t})) \in \mathbb{D}) \leq \|\mathbf{t}\|\, \mathbf{1}_\mathbb{D}(\mathbf{t}), \quad \mathbf{t} \in [0,\infty)^n.$$

Since the RHS is $\mathcal{T}$-integrable by (1.1.4), $\mathbf{m}_1$ is finite. Note that

$$\|\mathbb{E}[\mathbf{X}(\mathbf{t})\mathbf{1}_\mathbb{D}(\mathbf{t}, \mathbf{X}(\mathbf{t}))]\| \leq \|\mathbb{E}[(\mathbf{t}, \mathbf{X}(\mathbf{t}))\mathbf{1}_\mathbb{D}(\mathbf{t}, \mathbf{X}(\mathbf{t}))]\|, \quad \mathbf{t} \in [0,\infty)^n.$$

Then using (2.2.9) applied to the process $(I\mathbf{e}, \mathbf{X})$, followed by (1.1.4), we obtain the finiteness of the integral in (2.2.3), while the other terms are finite by Lemma 1.2.1. Thus, $\mathbf{m}_2$ is finite. So we have proved that $(\mathbf{m}, \Theta, \mathcal{Z})$ is a valid characteristic triplet for a $2n$-dimensional Lévy process.

(ii). On a suitable augmentation of $(\Omega, \mathcal{F}, \mathbb{P})$ on which $\mathbf{T}$ is defined, we find $\mathbf{W} \sim L^{2n}(\mathbf{m}, \Theta, \boldsymbol{\delta}_\mathbf{0} \otimes (\mathbf{d} \diamond \mathcal{X}))$ and a set $\xi = \{\xi(t, \mathbf{t}) : (t, \mathbf{t}) \in [0,\infty) \times [0,\infty)^n_*\}$ of independent random vectors satisfying $\xi(t,\mathbf{t}) \overset{D}{=} \mathbf{X}(\mathbf{t})$ for $(t,\mathbf{t}) \in [0,\infty) \times [0,\infty)^n_*$, such that $\mathbf{T}$, $\mathbf{W}$, ξ are independent.

The law of the jumps of $\mathbf{T}$ are determined by a Poisson random measure on $[0,\infty) \times [0,\infty)^n_*$ with intensity measure $\mathrm{d}t \otimes \mathcal{T}$ (see Chapter I, Theorem 1 in [Ber96]). The set $\{\xi(t, \Delta\mathbf{T}(t)) : t > 0,\ \Delta\mathbf{T}(t) \neq \mathbf{0}\}$ is countable, and the random vectors $\xi(t, \Delta\mathbf{T}(t))$ are conditionally independent given $(t, \Delta\mathbf{T}(t)) = (t, \mathbf{t})$ with distribution determined by the Markov kernel $(t, \mathbf{t}, A) \mapsto \mathbb{P}(\mathbf{X}(\mathbf{t}) \in A)$ for $(t, \mathbf{t}) \in [0, \infty) \times [0, \infty)^n_*$ and Borel sets $A \subseteq \mathbb{R}^n$. Thus, applying the marking theorem in Section 5.2 of [Kin93], we have that

$$\mathbb{Z}_0 := \sum_{t>0} \boldsymbol{\delta}_{(t, \Delta\mathbf{T}(t), \xi(t, \Delta\mathbf{T}(t)))}$$

is a Poisson random measure on $[0, \infty) \times [0, \infty)^n_* \times \mathbb{R}^n$ with intensity measure

$$\mathbb{P}(\mathbf{X}(\mathbf{t}) \in \mathrm{d}\mathbf{x}) \otimes (\mathrm{d}t \otimes \mathcal{T}(\mathrm{d}\mathbf{t})) = \mathrm{d}t \otimes \mathcal{Z}_0$$

by Fubini's theorem.

Given the Poisson random measure $\mathbb{Z}_0$, a Lévy process $\mathbf{Z}_0 \sim L^{2n}(\mathbf{0}, \mathbf{0}, \mathcal{Z}_0)$ with jump measure $\mathbb{Z}_0$ can be constructed through its sample paths using the Lévy-Itô decomposition (see Chapter VII, Theorem 1.29 in [Çın11]). Using the Lévy-Khintchine formula (1.1.2), we see that $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2) := \mathbf{Z}_0 + \mathbf{W} \overset{D}{=} (\mathbf{T}, \mathbf{X} \odot \mathbf{T})$.

Using the Lévy-Itô decomposition gives

$$\mathbf{Z}_1(t) = \sum_{s \in (0,t]} \Delta\mathbf{T}(s) - t \int_{\mathbb{R}^n_*} \mathbf{t}\mathbb{P}((\mathbf{t}, \mathbf{X}(\mathbf{t})) \in \mathbb{D})\, \mathcal{T}(\mathrm{d}\mathbf{t}) + t\mathbf{m}_1 = \mathbf{T}(t), \quad t \geq 0,$$

where $\mathbf{Z}_0$ contributes the first and second terms of the middle expression, and $\mathbf{W}$ contributes the third term. Thus, $\mathbf{Z}$ is the semi-strong subordination of $\mathbf{X}$ and $\mathbf{T}$. □

Remark 2.2.5. The proof of Theorem 2.2.4 (ii) can also be reframed in terms of marked Poisson point processes. The process $(t, \Delta\mathbf{T}(t))_{t>0, \Delta\mathbf{T}(t)\neq\mathbf{0}}$ is a Poisson point process on $[0, \infty) \times [0, \infty)^n_*$ with intensity measure $\mathrm{d}t \otimes \mathcal{T}$, $\{\xi(t, \Delta\mathbf{T}(t)) : t > 0,\ \Delta\mathbf{T}(t) \neq \mathbf{0}\}$ is a set of marks on the mark space $\mathbb{R}^n$, and we constructed the Lévy process $\mathbf{Z}_0$, which gives the jumps of the weakly subordinated process $\mathbf{Z}$ resulting from the jumps of $\mathbf{T}$, using the Poisson random measure $\mathbb{Z}_0$ that determines the law of the marked Poisson point process $(t, \Delta\mathbf{Z}(t))_{t>0, \Delta\mathbf{Z}(t)\neq\mathbf{0}}$ on $[0, \infty) \times [0, \infty)^n_* \times \mathbb{R}^n$.

Example 2.2.6. Recalling Example 1.3.5, $(I, 2I)$ is a subordinator and (B, B) is a Lévy process, but $(B, B) \circ (I, 2I)$ is not. Let B^* be a standard Brownian motion independent of B. From (2.2.1)–(2.2.5), the semi-strong subordination of (B, B) and $(I, 2I)$ is

$$\mathbf{Z} \sim L^4\left((1, 2, 0, 0), \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{d} \diamond \Sigma \end{pmatrix}, 0\right), \quad \mathbf{d} \diamond \Sigma = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Using Proposition 1.1.8 or the Lévy-Khintchine formula (1.1.2), we see that $(I, 2I, B,$

$B + B^*$) is a Lévy process with the same characteristic triplet. Thus,

$$\mathbf{Z} \stackrel{D}{=} ((I, 2I), (B, B) \odot (I, 2I)) \stackrel{D}{=} (I, 2I, B, B + B^*).$$

2.3 Properties of Weak Subordination

Now we prove a variety of useful properties of weak subordination.

Throughout this section, unless otherwise stated, we let $\mathbf{X} = (X_1, \dots, X_n) \sim L^n(\boldsymbol{\mu}, \Sigma, \mathcal{X})$ be a subordinate and $\mathbf{T} = (T_1, \dots, T_n) \sim S^n(\mathbf{d}, \mathcal{T})$ be a subordinator.

2.3.1 Characteristics

The next proposition gives a formula for the characteristic exponent of weakly subordinated processes, which can serve as an alternative definition. Recall that $\mathbf{d} \diamond \Psi_{\mathbf{X}}$ is defined in (1.2.7).

Proposition 2.3.1. *A process $\mathbf{Z} \stackrel{D}{=} (\mathbf{T}, \mathbf{X} \odot \mathbf{T})$ is the weak subordination of $\mathbf{X}$ and $\mathbf{T}$ if and only if $\mathbf{Z}$ has characteristic exponent*

$$\Psi_{\mathbf{Z}}(\boldsymbol{\theta}) = \mathrm{i}\langle \mathbf{d}, \boldsymbol{\theta}_1 \rangle + (\mathbf{d} \diamond \Psi_{\mathbf{X}})(\boldsymbol{\theta}_2) + \int_{[0,\infty)^n_*} (\Phi_{(\mathbf{t}, \mathbf{X}(\mathbf{t}))}(\boldsymbol{\theta}) - 1)\, \mathcal{T}(\mathrm{d}\mathbf{t}) \tag{2.3.1}$$

for all $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \mathbb{R}^n$.

Proof. Let $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \mathbb{R}^n$. Clearly, $\mathbf{Z} \stackrel{D}{=} (\mathbf{T}, \mathbf{X} \odot \mathbf{T})$ if and only if $\mathbf{Z} \sim L^{2n}(\mathbf{m}, \Theta, \mathcal{Z})$ has characteristic triplet as specified in (2.2.1)–(2.2.5). Using the Lévy-Khintchine formula (1.1.2), this occurs if and only if Z has characteristic exponent

$$\begin{aligned}\Psi_{\mathbf{Z}}(\boldsymbol{\theta}) = \mathrm{i}\langle \mathbf{m}, \boldsymbol{\theta} \rangle - \frac{1}{2}\|\boldsymbol{\theta}\|^2_{\Theta} + \int_{\mathbb{R}^n_*} (e^{\mathrm{i}\langle \boldsymbol{\theta}_2, \mathbf{x}\rangle} - 1 - \mathrm{i}\langle \boldsymbol{\theta}_2, \mathbf{x}\rangle \mathbf{1}_{\mathbb{D}}(\mathbf{x}))\, (\mathbf{d} \diamond \mathcal{X})(\mathrm{d}\mathbf{x}) \\ + I(\boldsymbol{\theta}),\end{aligned} \tag{2.3.2}$$

where

$$\begin{aligned} I(\boldsymbol{\theta}) &:= \int_{[0,\infty)^n_* \times \mathbb{R}^n} (e^{\mathrm{i}\langle \boldsymbol{\theta}, (\mathbf{t}, \mathbf{x})\rangle} - 1 - \mathrm{i}\langle \boldsymbol{\theta}, (\mathbf{t}, \mathbf{x})\rangle \mathbf{1}_{\mathbb{D}}(\mathbf{t}, \mathbf{x}))\, \mathbb{P}(\mathbf{X}(\mathbf{t}) \in \mathrm{d}\mathbf{x}) \mathcal{T}(\mathrm{d}\mathbf{t}) \\ &= -\mathrm{i} \int_{[0,\infty)^n_*} (\langle \boldsymbol{\theta}_1, \mathbf{t}\rangle \mathbb{P}((\mathbf{t}, \mathbf{X}(\mathbf{t})) \in \mathbb{D}) + \mathbb{E}[\langle \boldsymbol{\theta}_2, \mathbf{X}(\mathbf{t})\rangle \mathbf{1}_{\mathbb{D}}(\mathbf{t}, \mathbf{X}(\mathbf{t}))])\, \mathcal{T}(\mathrm{d}\mathbf{t}) \\ &\quad + \int_{[0,\infty)^n_*} (\Phi_{(\mathbf{t}, \mathbf{X}(\mathbf{t}))}(\boldsymbol{\theta}) - 1)\, \mathcal{T}(\mathrm{d}\mathbf{t}). \end{aligned} \tag{2.3.3}$$

Using (2.2.6) and then (1.1.4) yields the $\mathcal{T}$-integrability of $\mathbf{t} \mapsto \Phi_{(\mathbf{t},\mathbf{X}(\mathbf{t}))}(\boldsymbol{\theta}) - 1$ so that all the terms above are finite. Combining (2.3.2)–(2.3.3) and the identity $\|\boldsymbol{\theta}\|^2_{\Theta} = \|\boldsymbol{\theta}_2\|^2_{\mathbf{d}\diamond\Sigma}$ yields (2.3.1). □

Corollary 2.3.2. *Let $\boldsymbol{\alpha} \in [0,\infty)^n$ be a deterministic vector and $R \sim S^1(d,\mathcal{R})$ be a univariate subordinator. If $\mathbf{T} = R\boldsymbol{\alpha}$, then $\mathbf{Z} \stackrel{D}{=} (\mathbf{T}, \mathbf{X} \odot \mathbf{T})$ has characteristic exponent*

$$\Psi_{\mathbf{Z}}(\boldsymbol{\theta}) = \mathrm{i}d\langle \boldsymbol{\alpha}, \boldsymbol{\theta}_1\rangle + d(\boldsymbol{\alpha} \diamond \Psi_{\mathbf{X}})(\boldsymbol{\theta}_2) + \int_{(0,\infty)} \left(\Phi_{(r\boldsymbol{\alpha},\mathbf{X}(r\boldsymbol{\alpha}))}(\boldsymbol{\theta}) - 1\right) \mathcal{R}(\mathrm{d}r) \tag{2.3.4}$$

for all $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \mathbb{R}^n$.

Proof. By Proposition 1.1.8, $\mathbf{T} \sim S^n(d\boldsymbol{\alpha}, \mathcal{R} \circ (I\boldsymbol{\alpha})^{-1})$. Then, the result follows from Proposition 2.3.1 and the transformation theorem (see Proposition A.3.1). □

Now we determine the law of the projected process $\mathbf{Z}_2 \stackrel{D}{=} \mathbf{X} \odot \mathbf{T}$.

Remark 2.3.3. Note that $\mathbf{m}_2$ in (2.3.5) below is different from $\mathbf{m}_2$ in (2.2.3).

Proposition 2.3.4. *If $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2) \stackrel{D}{=} (\mathbf{T}, \mathbf{X} \odot \mathbf{T})$ is the weak subordination of $\mathbf{X}$ and $\mathbf{T}$, then $\mathbf{Z}_1 \stackrel{D}{=} \mathbf{T}$ and $\mathbf{Z}_2 \sim L^n(\mathbf{m}_2, \Theta_2, \mathcal{Z}_2)$ with*

$$\mathbf{m}_2 = \mathbf{c}(\mathbf{d}, \mathcal{X}) + \mathbf{d} \diamond \boldsymbol{\mu} + \int_{[0,\infty)^n_*} \mathbb{E}[\mathbf{X}(\mathbf{t})\mathbf{1}_{\mathbb{D}}(\mathbf{X}(\mathbf{t}))]\, \mathcal{T}(\mathrm{d}\mathbf{t}), \tag{2.3.5}$$

$$\Theta_2 = \mathbf{d} \diamond \Sigma, \tag{2.3.6}$$

$$\mathcal{Z}_2(\mathrm{d}\mathbf{x}) = \mathbf{d} \diamond \mathcal{X}(\mathrm{d}\mathbf{x}) + \int_{[0,\infty)^n_*} \mathbb{P}(\mathbf{X}(\mathbf{t}) \in \mathrm{d}\mathbf{x}) \mathcal{T}(\mathrm{d}\mathbf{t}). \tag{2.3.7}$$

Proof. Let $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \mathbb{R}^n$. Recall that $\mathbf{Z} \sim L^{2n}(\mathbf{m}, \Theta, \mathcal{Z})$ with $(\mathbf{m}, \Theta, \mathcal{Z})$ as specified in (2.2.1)–(2.2.5), and note Remark 2.3.3. We have $\Psi_{\mathbf{Z}}(\boldsymbol{\theta}_1, \mathbf{0}) = \Psi_{\mathbf{T}}(\boldsymbol{\theta}_1)$, where the LHS is computed using (2.3.1) and the RHS is given by (1.1.6), implying $\mathbf{Z}_1 \stackrel{D}{=} \mathbf{T}$. Likewise, $\mathbf{Z}_2 \sim L^n(\mathbf{m}_2, \Theta_2, \mathcal{Z}_2)$ with $(\mathbf{m}_2, \Theta_2, \mathcal{Z}_2)$ as specified in (2.3.5)–(2.3.7) because $\Psi_{\mathbf{Z}}(\mathbf{0}, \boldsymbol{\theta}_2) = \Psi_{\mathbf{Z}_2}(\boldsymbol{\theta}_2)$, where the LHS is computed using (2.3.1) and the RHS is computed using the Lévy-Khintchine formula (1.1.2). □

2.3.2 Consistency with Strong Subordination

Based on Proposition 1.3.2, strong subordination is known to produce a Lévy process when $\mathbf{T}$ has indistinguishable components or $\mathbf{X}$ has independent components. We now show that under these assumptions, their law coincides with that of weak and semi-strong subordination. Otherwise, based on Proposition 1.3.6, strong subordination may not always produce a Lévy process, while weak and semi-strong subordination always does by definition. In this sense, weak subordination is an extension of strong subordination.

Theorem 2.3.5. *Let* $\mathbf{X}$ *and* $\mathbf{T}$ *be independent. If* $\mathbf{T}$ *has indistinguishable components or* $\mathbf{X}$ *has independent components, then* $(\mathbf{T}, \mathbf{X} \circ \mathbf{T})$ *is the semi-strong subordination of* $\mathbf{X}$ *and* $\mathbf{T}$*, that is* $(\mathbf{T}, \mathbf{X} \circ \mathbf{T}) \stackrel{D}{=} (\mathbf{T}, \mathbf{X} \odot \mathbf{T})$.

Proof. Let $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \mathbb{R}^n$. Since $\mathbf{T}$ and $\mathbf{X}$ are independent processes, using Proposition 1.2.2 and conditioning on $\mathbf{T}$, we get

$$\Phi_{(\mathbf{T},\mathbf{X}\circ\mathbf{T})}(\boldsymbol{\theta}) = \mathbb{E}[\exp(\mathrm{i}\langle \boldsymbol{\theta}_1, \mathbf{T}(1)\rangle + (\mathbf{T}(1) \diamond \Psi_{\mathbf{X}})(\boldsymbol{\theta}_2))]. \tag{2.3.8}$$

The $2n$-dimensional Lévy process $(I\mathbf{e}, \mathbf{X})$ has independent components by assumption. By Proposition 1.1.8, $(\mathbf{T}, \mathbf{T})$ is a $2n$-dimensional Lévy process, in particular, a subordinator. Thus, $(I\mathbf{e}, \mathbf{X}) \circ (\mathbf{T}, \mathbf{T}) = (\mathbf{T}, \mathbf{X} \circ \mathbf{T})$ is a Lévy process by Proposition 1.3.2 (ii), so it suffices to show that $\Psi_{(\mathbf{T},\mathbf{X}\circ\mathbf{T})} = \Psi_{(\mathbf{T},\mathbf{X}\odot\mathbf{T})}$.

Univariate subordination. In this case, $\mathbf{T} = R\mathbf{e}$, where $R \sim S^1(d, \mathcal{R})$ and $\mathbf{e} = (1, \dots, 1) \in \mathbb{R}^n$. Let $z := -\mathrm{i}\langle \boldsymbol{\theta}_1, \mathbf{e}\rangle - (\mathbf{e} \diamond \Psi_{\mathbf{X}})(\boldsymbol{\theta}_2)$, which implies

$$-zr = \mathrm{i}\langle \boldsymbol{\theta}_1, r\mathbf{e}\rangle + ((r\mathbf{e}) \diamond \Psi_{\mathbf{X}})(\boldsymbol{\theta}_2) \tag{2.3.9}$$

for $r \geq 0$. Thus, (2.3.8) becomes $\Phi_{(\mathbf{T},\mathbf{X}\circ\mathbf{T})}(\boldsymbol{\theta}) = \mathbb{E}[\exp(-zR(1))]$. By noting that $\Re z \geq 0$ and applying Proposition 1.1.14, we have $\Psi_{(\mathbf{T},\mathbf{X}\circ\mathbf{T})}(\boldsymbol{\theta}) = -\Lambda_R(z)$, where

$$\Lambda_R(z) = dz + \int_{(0,\infty)} (1 - e^{-zr})\, \mathcal{R}(\mathrm{d}r).$$

Now using (2.3.9) and the fact that $e^{-zr} = \Phi_{(r\mathbf{e},\mathbf{X}(r))}(\boldsymbol{\theta})$ for $r > 0$, which is implied by (2.3.8), we have that $\Psi_{(\mathbf{T},\mathbf{X}\circ\mathbf{T})}(\boldsymbol{\theta})$ matches the RHS of (2.3.4) with $\boldsymbol{\alpha} = \mathbf{e}$. Therefore, by Corollary 2.3.2, $(\mathbf{T}, \mathbf{X} \circ \mathbf{T}) \stackrel{D}{=} (\mathbf{T}, \mathbf{X} \odot \mathbf{T})$ is the semi-strong subordination of $\mathbf{X}$ and $\mathbf{T}$.

Multivariate subordination. In this case, $X_1, \dots, X_n$ are independent. Recall that for $\emptyset \neq J \subseteq \{1, \dots, n\}$, $\mathbf{x}\boldsymbol{\pi}_J := \sum_{j\in J} x_j \mathbf{e}_j$, $\mathbf{x} \in \mathbb{R}^n$, and $\mathcal{X}_J := \mathcal{X} \circ \boldsymbol{\pi}_J^{-1}$. In particular, Σ is a diagonal matrix and $\mathcal{X} = \sum_{k=1}^n \mathcal{X}_{\{k\}}$ (see Exercise 12.10 in [Sat99]). Let

$$\mathbf{z} := \frac{1}{2}\boldsymbol{\theta}_2(\boldsymbol{\theta}_2 \diamond \Sigma) - \mathrm{i}(\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2 \diamond \boldsymbol{\mu}) - \sum_{k=1}^n \int_{\mathbb{R}^n_*} (e^{\mathrm{i}\langle \boldsymbol{\theta}_2, \mathbf{x}\rangle} - 1 - \mathrm{i}\langle \boldsymbol{\theta}_2, \mathbf{x}\rangle \mathbf{1}_{\mathbb{D}}(\mathbf{x}))\, \mathcal{X}_{\{k\}}(\mathrm{d}\mathbf{x})\mathbf{e}_k.$$

For $\emptyset \neq J \subseteq \{1, \dots, n\}$, $1 \leq k \leq n$, note that $\mathbf{c} = \mathbf{0}$ in (1.2.3) because

$$\int_{\mathbb{D}^C} \mathbf{x}\boldsymbol{\pi}_J \mathbf{1}_{\mathbb{D}}(\mathbf{x}\boldsymbol{\pi}_J)\, \mathcal{X}_{\{k\}}(\mathrm{d}\mathbf{x}) = \mathbf{1}_J(k) \int_{\mathbb{R}^n_*} \mathbf{1}_{\mathbb{D}^C}(\mathbf{x}\boldsymbol{\pi}_{\{k\}})\mathbf{x}\boldsymbol{\pi}_{\{k\}}\mathbf{1}_{\mathbb{D}}(\mathbf{x}\boldsymbol{\pi}_{\{k\}})\, \mathcal{X}(\mathrm{d}x) = \mathbf{0}.$$

Let $\mathbf{t} = (t_1, \dots, t_n) \in [0, \infty)^n$. Recalling that Σ is diagonal, $\langle \boldsymbol{\theta}_2(\boldsymbol{\theta}_2 \diamond \Sigma), \mathbf{t}\rangle = \|\boldsymbol{\theta}_2\|^2_{\mathbf{t}\diamond\Sigma}$. If $J_{(m)} := \{(m), \dots, (n)\}$, $1 \leq m \leq n$, then (1.2.2) becomes

$$\begin{aligned}
\mathbf{t} \diamond \mathcal{X} &= \sum_{m=1}^{n} \Delta t_{(m)} \left(\sum_{k=1}^{n} \mathcal{X}_{\{k\}} \right)_{J_{(m)}} \\
&= \sum_{m=1}^{n} \Delta t_{(m)} \left(\sum_{k=1}^{n} \mathbf{1}_{J_{(m)}}(k) \mathcal{X}_{\{k\}} \right) \\
&= \sum_{k=1}^{n} \left(\sum_{m=1}^{n} \Delta t_{(m)} \mathbf{1}_{J_{(m)}}(k) \right) \mathcal{X}_{\{k\}} \\
&= \sum_{k=1}^{n} t_k \mathcal{X}_{\{k\}}.
\end{aligned}$$

Combining the results of the three previous sentences yields

$$-\langle \mathbf{z}, \mathbf{t}\rangle = \mathrm{i}\langle \boldsymbol{\theta}_1, \mathbf{t}\rangle + (\mathbf{t} \diamond \Psi_X)(\boldsymbol{\theta}_2) \tag{2.3.10}$$

for $\mathbf{t} \in [0, \infty)^n$. Thus, (2.3.8) becomes $\Phi_{(\mathbf{T},\mathbf{X}\circ\mathbf{T})}(\boldsymbol{\theta}) = \mathbb{E}[\exp(-\langle \mathbf{z}, \mathbf{T}(1)\rangle)]$. By noting that $\Re\mathbf{z} \in [0, \infty)^n$ and applying Proposition 1.1.14, we have $\Psi_{(\mathbf{T},\mathbf{X}\circ\mathbf{T})}(\boldsymbol{\theta}) = -\Lambda_{\mathbf{T}}(\mathbf{z})$, where

$$\Lambda_{\mathbf{T}}(\mathbf{z}) = \langle \mathbf{d}, \mathbf{z}\rangle + \int_{[0,\infty)^n_*} (1 - e^{-\langle \mathbf{z},\mathbf{t}\rangle})\, \mathcal{T}(\mathrm{d}\mathbf{t}).$$

Now using (2.3.10) and the fact that $e^{-\langle \mathbf{z},\mathbf{t}\rangle} = \Phi_{(\mathbf{t},\mathbf{X}(\mathbf{t}))}(\boldsymbol{\theta})$ for $\mathbf{t} \in [0, \infty)^n_*$, which is implied by (2.3.8), we have that $\Psi_{(\mathbf{T},\mathbf{X}\circ\mathbf{T})}(\boldsymbol{\theta})$ matches the RHS of (2.3.1). Therefore, by Proposition 2.3.1, $(\mathbf{T}, \mathbf{X} \circ \mathbf{T}) \stackrel{D}{=} (\mathbf{T}, \mathbf{X} \odot \mathbf{T})$ is the semi-strong subordination of $\mathbf{X}$ and $\mathbf{T}$. □

2.3.3 Linear Transformations

In this subsection, we discuss how linear transformations affect weakly subordinated processes. In particular, we show that weak subordination, like traditional subordination, is consistent with projections and permutations.

Proposition 2.3.6. *Let $A \in \mathbb{R}^{n\times n}$ be such that $\mathbf{T}A$ is a subordinator. If $\mathbf{X}(\mathbf{t})A \stackrel{D}{=} (\mathbf{X}A)(\mathbf{t}A)$ for all $\mathbf{t} \in [0, \infty)^n$ in the support of $\mathcal{T}$, then $(\mathbf{T}A, (\mathbf{X} \odot \mathbf{T})A) \stackrel{D}{=} (\mathbf{T}A, (\mathbf{X}A) \odot (\mathbf{T}A))$. If $\mathbf{T} = I\mathbf{d}$ for some $\mathbf{d} \in [0, \infty)^n$, then the converse also holds.*

Proof. Let $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \mathbb{R}^n$. We make repeated use of the fact that $\langle \mathbf{x}A, \mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{y}A'\rangle$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. We have

$$\begin{aligned} \Psi_{(\mathbf{T}A,(\mathbf{X}\odot\mathbf{T})A)}(\boldsymbol{\theta}) &= \Psi_{(\mathbf{T},\mathbf{X}\odot\mathbf{T})}(\boldsymbol{\theta}_1 A', \boldsymbol{\theta}_2 A') \\ &= \mathrm{i}\langle \mathbf{d}, \boldsymbol{\theta}_1 A'\rangle + (\mathbf{d} \diamond \Psi_{\mathbf{X}})(\boldsymbol{\theta}_2 A') \\ &\quad + \int_{[0,\infty)^n_*} (\Phi_{(\mathbf{t},\mathbf{X}(\mathbf{t}))}(\boldsymbol{\theta}_1 A', \boldsymbol{\theta}_2 A') - 1)\, \mathcal{T}(\mathrm{d}\mathbf{t}), \end{aligned} \tag{2.3.11}$$

where the first equality follows from (1.1.1) and the second equality is due to (2.3.1).

Sufficiency. The first term of (2.3.11) satisfies $\mathrm{i}\langle \mathbf{d}, \boldsymbol{\theta}_1 A'\rangle = \mathrm{i}\langle \mathbf{d}A, \boldsymbol{\theta}_1\rangle$. The second term satisfies $\mathbf{d} \diamond \Psi_{\mathbf{X}}(\boldsymbol{\theta}_2 A') = (\mathbf{d}A) \diamond \Psi_{\mathbf{X}A}(\boldsymbol{\theta}_2)$ as a result of Proposition 1.2.2 and the assumption implying $\langle \boldsymbol{\theta}_2 A', \mathbf{X}(\mathbf{d})\rangle = \langle \boldsymbol{\theta}_2, (\mathbf{X}A)(\mathbf{d}A)\rangle$. The third term satisfies

$$\int_{[0,\infty)^n_*} (\Phi_{(\mathbf{t},\mathbf{X}(\mathbf{t}))}(\boldsymbol{\theta}_1 A', \boldsymbol{\theta}_2 A') - 1)\, \mathcal{T}(\mathrm{d}\mathbf{t}) = \int_{[0,\infty)^n_*} (\Phi_{(\mathbf{t},(\mathbf{X}A)(\mathbf{t}))}(\boldsymbol{\theta}) - 1)\, (\mathcal{T} \circ A^{-1})(\mathrm{d}\mathbf{t}),$$

as a result of the transformation theorem (see Proposition A.3.1) and the assumption implying

$$\langle (\boldsymbol{\theta}_1 A', \boldsymbol{\theta}_2 A'), (\mathbf{t}, \mathbf{X}(\mathbf{t}))\rangle = \langle (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2), (\mathbf{t}A, (\mathbf{X}A)(\mathbf{t}A))\rangle$$

for all $\mathbf{t} \in [0,\infty)^n$ in the support of $\mathcal{T}$. Thus, by noting that $\mathbf{T}A \sim S^n(\mathbf{d}A, \mathcal{T} \circ A^{-1})$ as a result of Proposition 1.1.8, (2.3.11) equals $\Psi_{(\mathbf{T}A,(\mathbf{X}A)\odot(\mathbf{T}A))}(\boldsymbol{\theta})$ as determined by (2.3.1).

Necessity. By assumption, we can equate (2.3.11) with $\Psi_{(\mathbf{T}A,(\mathbf{X}A)\odot(\mathbf{T}A))}(\boldsymbol{\theta})$ to obtain

$$\mathrm{i}\langle \mathbf{d}, \boldsymbol{\theta}_1 A'\rangle + (\mathbf{d} \diamond \Psi_{\mathbf{X}})(\boldsymbol{\theta}_2 A') = \mathrm{i}\langle \mathbf{d}A, \boldsymbol{\theta}_1\rangle + (\mathbf{d}A) \diamond \Psi_{\mathbf{X}A}(\boldsymbol{\theta}_2),$$

which implies

$$\mathbb{E}[\exp(\mathrm{i}\langle \boldsymbol{\theta}_2, \mathbf{X}(t\mathbf{d})A\rangle)] = \mathbb{E}[\exp(\mathrm{i}\langle \boldsymbol{\theta}_2, (\mathbf{X}A)(t\mathbf{d}A)\rangle)], \quad t \geq 0.$$

Thus, $\mathbf{X}(\mathbf{t})A \overset{D}{=} (\mathbf{X}A)(\mathbf{t}A)$ for all $\mathbf{t} \in \{t\mathbf{d} : t \geq 0\}$ as required. □

An immediate corollary is that weak subordination is consistent with projections and permutations, and satisfies a *marginal component consistency* property. Let $\mathbf{X} \odot \mathbf{T} = ((\mathbf{X} \odot \mathbf{T})_1, \dots, (\mathbf{X} \odot \mathbf{T})_n)$, and recall that $\mathbf{x}\boldsymbol{\pi}_J := \sum_{j\in J} x_j \mathbf{e}_j$ for $\emptyset \neq J \subseteq \{1, \dots, n\}$ with $\boldsymbol{\pi}_\emptyset \equiv \mathbf{0}$.

Corollary 2.3.7. *If $J \subseteq \{1, \dots, n\}$, then $(\mathbf{T}\boldsymbol{\pi}_J, (\mathbf{X} \odot \mathbf{T})\boldsymbol{\pi}_J) \stackrel{D}{=} (\mathbf{T}\boldsymbol{\pi}_J, (\mathbf{X}\boldsymbol{\pi}_J) \odot (\mathbf{T}\boldsymbol{\pi}_J))$. In particular, $(T_k, (\mathbf{X} \odot \mathbf{T})_k) \stackrel{D}{=} (T_k, X_k \odot T_k)$ for $1 \le k \le n$.*

If, in addition, $\mathbf{T}$ and $\mathbf{X}$ are independent, then $(T_k, (\mathbf{X} \odot \mathbf{T})_k) \stackrel{D}{=} (T_k, X_k \circ T_k)$ for $1 \le k \le n$.

Proof. The first statement immediately follows from Proposition 2.3.6 with $A = \boldsymbol{\pi}_J$ since $\mathbf{X}(\mathbf{t})\boldsymbol{\pi}_J \stackrel{D}{=} (\mathbf{X}\boldsymbol{\pi}_J)(\mathbf{t}\boldsymbol{\pi}_J)$ for all $\mathbf{t} \in [0, \infty)^n$. The second statement is the special case of $J = \{k\}$. The last statement follows from Theorem 2.3.5. □

Corollary 2.3.8. *Let $P \in \mathbb{R}^{n \times n}$ be a permutation matrix, then $(\mathbf{T}P, (\mathbf{X} \odot \mathbf{T})P) \stackrel{D}{=} (\mathbf{T}P, (\mathbf{X}P) \odot (\mathbf{T}P))$.*

Proof. This immediately follows from Proposition 2.3.6 with $A = P$ since $\mathbf{X}(\mathbf{t})P \stackrel{D}{=} (\mathbf{X}P)(\mathbf{t}P)$ for all $\mathbf{t} \in [0, \infty)^n$. □

For $A \in \mathbb{R}^{n \times n}$, $\mathbf{X} \sim L^n(\boldsymbol{\mu}, \Sigma, \mathcal{X})$ and $\mathbf{T} \sim S^n(\mathbf{d}, \mathcal{T})$, it may also be natural to ask whether the law of $(\mathbf{X} \odot \mathbf{T})A$, is equal to the law of $\mathbf{X}^* \odot \mathbf{T}^*$, $(\mathbf{X}A) \odot \mathbf{T}^*$ or $\mathbf{X}^* \odot (\mathbf{T}A)$ for some subordinate $\mathbf{X}^* \sim L^n(\boldsymbol{\mu}^*, \Sigma^*, \mathcal{X}^*)$ and subordinator $\mathbf{T}^* \sim S^n(\mathbf{d}^*, \mathcal{T}^*)$. We address these questions in Remarks 2.3.9–2.3.11, respectively.

Remark 2.3.9. Let $A \in \mathbb{R}^{n \times n}$, then $(\mathbf{X} \odot \mathbf{T})A \stackrel{D}{=} \mathbf{X}^* \odot \mathbf{T}^*$ holds trivially for the subordinate $\mathbf{X}^* \stackrel{D}{=} (\mathbf{X} \odot \mathbf{T})A$ and the subordinator $\mathbf{T}^* = I\mathbf{e}$. So every linearly transformed strongly subordinated process with the subordinate having independent components, or in fact any Lévy process, is a weakly subordinated process. This property does not hold in general for strong subordination.

Remark 2.3.10. If $A \in \mathbb{R}^{n \times n}$ is a projection or permutation matrix, then by Proposition 2.3.6, $(\mathbf{X} \odot \mathbf{T})A \stackrel{D}{=} (\mathbf{X}A) \odot \mathbf{T}^*$ holds for $\mathbf{T}^* \stackrel{D}{=} \mathbf{T}A$.

However, there are cases where this fails. Suppose that $\mathbf{X} \sim BM^2(\mathbf{0}, \Sigma)$ and $\mathbf{T} = I\mathbf{d}$, where

$$A = \Sigma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{d} = (1, 2). \tag{2.3.12}$$

Since $(\mathbf{X} \odot \mathbf{T})A$ has continuous sample paths a.s., in order for $(\mathbf{X} \odot \mathbf{T})A \stackrel{D}{=} (\mathbf{X}A) \odot \mathbf{T}^*$ to hold, we must have $\mathbf{T}^* \sim S^2(\mathbf{d}^*, 0)$ for some $\mathbf{d}^* = (d_1^*, d_2^*) \in [0, \infty)^2$. By equating the matrix component of the characteristic triplet of $(\mathbf{X} \odot \mathbf{T})A$ and $(\mathbf{X}A) \odot \mathbf{T}^*$, calculated using Propositions 1.1.8 and 2.3.4, we require $A'(\mathbf{d} \diamond \Sigma)A = \mathbf{d}^* \diamond (A'\Sigma A)$. For the choices in (2.3.12), this implies

$$d_1^* = \frac{14}{13}, \quad d_2^* = \frac{6}{5}, \quad d_1^* \wedge d_2^* = \frac{9}{8},$$

which is impossible to satisfy for any $\mathbf{d}^* \in [0,\infty)^2$. In summary, there does not always exist $\mathbf{X}^* \sim L^n(\boldsymbol{\mu}^*, \Sigma^*, \mathcal{X}^*)$ and $\mathbf{T}^* \sim S^n(\mathbf{d}^*, \mathcal{T}^*)$ such that $(\mathbf{X} \odot \mathbf{T})A \overset{D}{=} (\mathbf{X}A) \odot \mathbf{T}^*$.

Remark 2.3.11. If $A \in \mathbb{R}^{n\times n}$ is a projection or permutation matrix, then by Proposition 2.3.6, $(\mathbf{X} \odot \mathbf{T})A \overset{D}{=} \mathbf{X}^* \odot (\mathbf{T}A)$ holds for $\mathbf{X}^* \overset{D}{=} \mathbf{X}A$.

However, there are again cases where this fails. Let $\mathbf{X} \sim BM^2(\mathbf{0}, \Sigma)$, $\mathbf{T} \sim S^2(0, \boldsymbol{\delta}_{\mathbf{d}})$ with A, Σ, $\mathbf{d}$ defined in (2.3.12). For $\boldsymbol{\theta} \in \mathbb{R}^n$, using (2.3.11), we have $\Psi_{(\mathbf{X}\odot\mathbf{T})A}(\boldsymbol{\theta}) = \Phi_{\mathbf{X}(\mathbf{d})A}(\boldsymbol{\theta}) - 1$. Note that $\mathbf{T}A \sim S^2(0, \boldsymbol{\delta}_{\mathbf{d}A})$, so $\Psi_{\mathbf{X}^*\odot(\mathbf{T}A)}(\boldsymbol{\theta}) = \Phi_{\mathbf{X}^*(\mathbf{d}A)}(\boldsymbol{\theta}) - 1$ using (2.3.1). Now in order for $(\mathbf{X} \odot \mathbf{T})A \overset{D}{=} \mathbf{X}^* \odot (\mathbf{T}A)$ to hold, we must have $\Phi_{\mathbf{X}(\mathbf{d})A}(\boldsymbol{\theta}) = \Phi_{\mathbf{X}^*(\mathbf{d}A)}(\boldsymbol{\theta})$. In particular, the matrix component of the characteristic triplet of $\mathbf{X}(\mathbf{d})A$ and $\mathbf{X}^*(\mathbf{d}A)$ must be equal, giving $A'(\mathbf{d} \diamond \Sigma)A = (\mathbf{d}A) \diamond \Sigma^*$. These values of A, Σ and $\mathbf{d}$ imply that

$$\Sigma^* = \begin{pmatrix} 7/2 & 3 \\ 3 & 2 \end{pmatrix},$$

which is not a covariance matrix. In summary, there does not always exist $\mathbf{X}^* \sim L^n(\boldsymbol{\mu}^*, \Sigma^*, \mathcal{X}^*)$ and $\mathbf{T}^* \sim S^n(\mathbf{d}^*, \mathcal{T}^*)$ such that $(\mathbf{X} \odot \mathbf{T})A \overset{D}{=} \mathbf{X}^* \odot (\mathbf{T}A)$.

Remark 2.3.12. The converse in Proposition 2.3.6 may fail without the assumption that $\mathbf{T} = I\mathbf{d}$ as the counterexample in Remark 2.3.11 demonstrates.

2.3.4 Ray Subordination and Superposition of Subordinators

If $\boldsymbol{\alpha} \in [0,\infty)^n$ is a deterministic vector and R is a univariate subordinator, then $\mathbf{T} = R\boldsymbol{\alpha}$ defines an n-dimensional *ray subordinator* travelling along the deterministic ray $\{r\boldsymbol{\alpha} : r \geq 0\}$. We refer to subordination with $\mathbf{T}$ as *ray subordination*. A special case is univariate subordination where the corresponding ray is $\{r\mathbf{e} : r \geq 0\}$ with $\mathbf{e} = (1, \ldots, 1) \in \mathbb{R}^n$.

Proposition 2.3.13 shows that weak subordination with a ray subordinator can be viewed as univariate subordination of an augmented process. Proposition 2.3.15 shows that weak subordination with a superposition of independent subordinators has law coinciding with that of a superposition of independent weakly subordinated processes.

Proposition 2.3.13. *Let $\boldsymbol{\alpha} \in [0,\infty)^n$ be a deterministic vector, R be a univariate subordinator and $\mathbf{Y}$ be a Lévy process with characteristic exponent $\Psi_{\mathbf{Y}} = \boldsymbol{\alpha} \diamond \Psi_{\mathbf{X}}$. Then $(R\boldsymbol{\alpha}, \mathbf{X} \odot (R\boldsymbol{\alpha})) \overset{D}{=} (I\boldsymbol{\alpha}, \mathbf{Y}) \odot (R(\mathbf{e}, \mathbf{e}))$.*

If, in addition, R and $\mathbf{Y}$ are independent, then $(R\boldsymbol{\alpha}, \mathbf{X} \odot (R\boldsymbol{\alpha})) \overset{D}{=} (R\boldsymbol{\alpha}, \mathbf{Y} \circ (R\mathbf{e}))$.

Proof. Let $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \mathbb{R}^n$. Suppose $R \sim S^1(d, \mathcal{R})$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in [0, \infty)^n$. Denote the augmented process by $\mathbf{W} := (I\boldsymbol{\alpha}, \mathbf{Y})$. Proposition 1.2.2 implies that

$$\mathbf{W}(r) = (r\boldsymbol{\alpha}, \mathbf{Y}(r)) \stackrel{D}{=} (r\boldsymbol{\alpha}, \mathbf{X}(r\boldsymbol{\alpha})), \quad r \geq 0. \tag{2.3.13}$$

which gives $I_1(\boldsymbol{\theta}) = I_2(\boldsymbol{\theta})$, where

$$I_1(\boldsymbol{\theta}) := \int_{(0,\infty)} (\Phi_{(r\boldsymbol{\alpha}, \mathbf{X}(r\boldsymbol{\alpha}))}(\boldsymbol{\theta}) - 1)\, \mathcal{R}(\mathrm{d}r),$$
$$I_2(\boldsymbol{\theta}) := \int_{(0,\infty)} (\Phi_{\mathbf{W}(r(\mathbf{e},\mathbf{e}))}(\boldsymbol{\theta}) - 1)\, \mathcal{R}(\mathrm{d}r).$$

Using Corollary 2.3.2 and recalling $\boldsymbol{\alpha} \diamond \Psi_{\mathbf{X}}(\boldsymbol{\theta}_2) = \Psi_{\mathbf{Y}}(\boldsymbol{\theta}_2)$, we get

$$\Psi_{(R\boldsymbol{\alpha}, \mathbf{X} \odot (R\boldsymbol{\alpha}))}(\boldsymbol{\theta}) = \mathrm{i}d\langle \boldsymbol{\alpha}, \boldsymbol{\theta}_1 \rangle + d\Psi_{\mathbf{Y}}(\boldsymbol{\theta}_2) + I_1(\boldsymbol{\theta}).$$

Next, observe that $\Psi_{\mathbf{W} \odot (R(\mathbf{e},\mathbf{e}))}(\boldsymbol{\theta}) = \Psi_{(R(\mathbf{e},\mathbf{e}), \mathbf{W} \odot (R(\mathbf{e},\mathbf{e})))}(\mathbf{0}, \boldsymbol{\theta})$. By Corollary 2.3.2 and then (2.3.13), the RHS evaluates to

$$d(\mathbf{e}, \mathbf{e}) \diamond \Psi_{\mathbf{W}}(\boldsymbol{\theta}) + I_2(\boldsymbol{\theta}) = \mathrm{i}d\langle \boldsymbol{\alpha}, \boldsymbol{\theta}_1 \rangle + d\Psi_{\mathbf{Y}}(\boldsymbol{\theta}_2) + I_2(\boldsymbol{\theta}).$$

Thus, $\Psi_{(R\boldsymbol{\alpha}, \mathbf{X} \odot (R\boldsymbol{\alpha}))}(\boldsymbol{\theta}) = \Psi_{\mathbf{W} \odot (R(\mathbf{e},\mathbf{e}))}(\boldsymbol{\theta})$.

If R and $\mathbf{Y}$ are independent, then $(R\boldsymbol{\alpha}, \mathbf{X} \odot (R\boldsymbol{\alpha})) \stackrel{D}{=} (I\boldsymbol{\alpha}, \mathbf{Y}) \circ (R(\mathbf{e}, \mathbf{e}))$ by Theorem 2.3.5, from which the last statement follows. □

Example 2.3.14. Let B, B^*, N be independent processes, where B, B^* are standard Brownian motions and N is a Poisson process with unit rate. From Example 1.2.3, the Lévy process with characteristic exponent $(1, 2) \diamond \Psi_{(B,B)}$ is $(B, B + B^*)$. Thus, by Proposition 2.3.13,

$$((I, 2I), (B, B) \odot (I, 2I)) \stackrel{D}{=} (I, 2I, B, B + B^*) \circ (I, I, I, I),$$
$$((N, 2N), (B, B) \odot (N, 2N)) \stackrel{D}{=} (I, 2I, B, B + B^*) \circ (N, N, N, N).$$

So we can represent these weakly subordinated processes using univariate subordination.

Proposition 2.3.15. *Let* $\mathbf{d} \in [0, \infty)^n$ *and* $\mathbf{T}^{(1)}, \dots, \mathbf{T}^{(m)}$ *be independent* n*-dimensional driftless subordinators with* $\mathbf{T} \stackrel{D}{=} I\mathbf{d} + \sum_{k=1}^m \mathbf{T}^{(k)}$*. Then* $(\mathbf{T}, \mathbf{X} \odot \mathbf{T}) \stackrel{D}{=} \sum_{k=0}^m \mathbf{A}^{(k)}$*, where* $\mathbf{A}^{(0)}, \mathbf{A}^{(1)}, \dots, \mathbf{A}^{(m)}$ *are independent Lévy processes with* $\mathbf{A}^{(0)} \stackrel{D}{=} (I\mathbf{d}, \mathbf{X} \odot I\mathbf{d})$ *and* $\mathbf{A}^{(k)} \stackrel{D}{=} (\mathbf{T}^{(k)}, \mathbf{X} \odot \mathbf{T}^{(k)})$*,* $1 \leq k \leq m$*.*

Proof. Assume that $\mathbf{T}^{(1)}, \dots, \mathbf{T}^{(m)}, \mathbf{A}^{(0)}, \dots, \mathbf{A}^{(m)}$ are independent processes, where $\mathbf{T}^{(k)} \sim S^n(\mathbf{0}, \mathcal{T}_k)$, $1 \leq k \leq m$, so that $\mathbf{T} \sim S^n(\mathbf{d}, \sum_{k=1}^m \mathcal{T}_k)$. By (2.3.1) and the independence of $\mathbf{A}^{(0)}, \dots, \mathbf{A}^{(m)}$, we have

$$\begin{aligned}\Psi_{(\mathbf{T},\mathbf{X}\odot\mathbf{T})}(\boldsymbol{\theta}) &= \mathrm{i}\langle \mathbf{d}, \boldsymbol{\theta}_1\rangle + (\mathbf{d} \diamond \Psi_{\mathbf{X}})(\boldsymbol{\theta}_2) + \int_{[0,\infty)^n_*} (\Phi_{(\mathbf{t},\mathbf{X}(\mathbf{t}))}(\boldsymbol{\theta}) - 1) \left(\sum_{k=1}^m \mathcal{T}_k\right)(\mathrm{d}\mathbf{t}) \\ &= \sum_{k=0}^m \Psi_{\mathbf{A}^{(k)}}(\boldsymbol{\theta}) \\ &= \Psi_{\sum_{k=0}^m \mathbf{A}^{(k)}}(\boldsymbol{\theta})\end{aligned}$$

for $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \mathbb{R}^n$, as required. □

In the context of strong subordination, Proposition 2.3.15 holds without assuming the subordinators $\mathbf{T}^{(1)}, \dots, \mathbf{T}^{(m)}$ are driftless (see Proposition 4.3 in [BKMS17]). However, this does not extend in general to weak subordination as the next example shows.

Example 2.3.16. Let B, B^*, W, W^* be independent standard Brownian motions. Example 2.2.6 states that $(B, B) \odot (I, 2I) \stackrel{D}{=} (B, B + B^*)$ and $(B, B) \odot (2I, I) \stackrel{D}{=} (W + W^*, W)$. Theorem 2.3.5 implies that $(B, B) \odot (3I, 3I) \stackrel{D}{=} (B, B) \circ (3I, 3I)$. However,

$$\begin{aligned}(B, B + B^*) + (W + W^*, W) &\sim BM^2\left(\mathbf{0}, \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}\right), \\ (B, B) \circ (3I, 3I) &\sim BM^2\left(\mathbf{0}, \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\right).\end{aligned}$$

So the conclusion of Proposition 2.3.15 cannot hold for the subordinate (B, B) and the subordinators $(I, 2I)$ and $(2I, I)$.

Remark 2.3.17. Subordinators are often formed by a superposition of independent ray subordinators. Some examples can be found in Section 2.5 of [BKMS17], Section 3 of [LS10] and Section 4.4 here. In these situations, Propositions 2.3.13 and 2.3.15, can be used to determine the law of weakly subordinated processes.

2.3.5 Subordinators with Independent Components

The next proposition deals with the weak subordination of driftless subordinators having independent components.

Proposition 2.3.18. *If* $\mathbf{T}$ *is a driftless subordinator with independent components, then* $\mathbf{X} \odot \mathbf{T}$ *also has independent components.*

Proof. If $\mathbf{T} \sim S^n(\mathbf{0}, \mathcal{T})$ has independent components $T_1 \sim S^1(0, \mathcal{T}_1), \dots, T_n \sim S^1(0, \mathcal{T}_n)$, then

$$\mathcal{T} = \sum_{k=1}^{n} \boldsymbol{\delta}_0^{\otimes(k-1)} \otimes \mathcal{T}_k \otimes \boldsymbol{\delta}_0^{\otimes(n-k)} \tag{2.3.14}$$

(see Exercise 12.10 in [Sat99]). By combining Corollary 2.3.7 and Proposition 1.3.2, $(\mathbf{X} \odot \mathbf{T})_k$, $1 \le k \le n$, has Lévy measure $\mathcal{Y}_k(\mathrm{d}x) = \int_{(0,\infty)} \mathbb{P}(X_k(t) \in \mathrm{d}x)\, \mathcal{T}_k(\mathrm{d}t)$. Thus, by substituting (2.3.14) into (2.3.7) we get,

$$\mathcal{Z}_2(\mathrm{d}\mathbf{x}) = \sum_{k=1}^{n} \int_{(0,\infty)} \mathbb{P}(\mathbf{X}(\mathbf{t}\boldsymbol{\pi}_{\{k\}}) \in \mathrm{d}\mathbf{x})\, \mathcal{T}_k(\mathrm{d}t_k) = \left(\sum_{k=1}^{n} \boldsymbol{\delta}_0^{\otimes(k-1)} \otimes \mathcal{Y}_k \otimes \boldsymbol{\delta}_0^{\otimes(n-k)} \right)(\mathrm{d}\mathbf{x}),$$

where $\mathbf{t} = (t_1, \dots, t_n) \in [0,\infty)^n_*$. Also, in (2.3.6), $\Theta_2 = 0$. Hence, $\mathbf{X} \odot \mathbf{T}$ has independent components. □

Remark 2.3.19. In general, the assumption that $\mathbf{T}$ is driftless in Proposition 2.3.18 cannot be dropped as that could allow $\Theta_2 = \mathbf{d} \diamond \Sigma$ in the proof to be nonzero when $\mathbf{T}$ has nonzero drift $\mathbf{d}$. Then $\mathbf{X} \odot \mathbf{T}$ cannot have independent components.

Example 2.3.20. Let B, B^*, N, N^* be independent processes, where B, B^* are standard Brownian motions and N, N^* are Poisson processes with unit rate. While the subordinate (B, B) has identical components, the weakly subordinated process $(B, B) \odot (N, N^*) \stackrel{D}{=} (B \circ N, B^* \circ N^*)$ has independent components by applying Proposition 2.3.18 and then Theorem 2.3.5 on each component.

2.3.6 Sample Path Properties

The following proposition gives a criterion for a weakly subordinated process to have finite variation in terms of a q-variation condition on its subordinator. The q-variation of a Lévy process with Lévy measure $\mathcal{X}$ is related to the Blumenthal-Getoor index $\inf_{q \ge 0}\{\int_{\mathbb{D}_*} \|\mathbf{x}\|^q\, \mathcal{X}(\mathrm{d}\mathbf{x}) < \infty\} \in [0,2]$, which measures the jump activity of the process [ASJ12, BG61]. When this index is no more than 1, the Lévy process has finite variation.

Recall $\mathbb{D}_*^+ := \mathbb{D} \cap [0,\infty)^n_*$.

Proposition 2.3.21. *Let* $\mathbf{Z} \stackrel{D}{=} (\mathbf{T}, \mathbf{X} \odot \mathbf{T})$. *If* $\mathbf{d} = \mathbf{0}$ *and* $\int_{\mathbb{D}_*^+} \|\mathbf{t}\|^{1/2}\, \mathcal{T}(\mathrm{d}\mathbf{t}) < \infty$, *then* $\mathbf{Z} \sim FV^{2n}$ *and driftless.*

Proof. Assuming $\mathbf{d} = \mathbf{0}$, we have $\Theta = 0$, and also the Lévy measure of $\mathbf{Z}$ reduces to $\mathcal{Z}_0$ as defined in (2.2.14), so we need to check that $\int_{\mathbb{D}_*} \|(\mathbf{t}, \mathbf{x})\| \, \mathcal{Z}_0(\mathrm{d}\mathbf{t}, \mathrm{d}\mathbf{x}) < \infty$. This integral is

$$\begin{aligned}\int_{[0,\infty)^n_*} \int_{\mathbb{R}^n} 1 \wedge \|(\mathbf{t}, \mathbf{x})\| \, \mathbb{P}(\mathbf{X}(\mathbf{t}) \in \mathrm{d}\mathbf{x}) \mathcal{T}(\mathrm{d}\mathbf{t}) &= \int_{[0,\infty)^n_*} \mathbb{E}[1 \wedge \|(\mathbf{t}, \mathbf{X}(\mathbf{t}))\|] \, \mathcal{T}(\mathrm{d}\mathbf{t}) \\ &\le C_2^{1/2} \int_{[0,\infty)^n_*} 1 \wedge \|(\mathbf{t}, \mathbf{t})\|^{1/2} \, \mathcal{T}(\mathrm{d}\mathbf{t}) \\ &\le 2^{1/4} C_2^{1/2} \int_{[0,\infty)^n_*} 1 \wedge \|\mathbf{t}\|^{1/2} \, \mathcal{T}(\mathrm{d}\mathbf{t}),\end{aligned}$$

where we have used (2.2.8) for the process $(I\mathbf{e}, \mathbf{X})$. The RHS is finite by the assumption $\int_{\mathbb{D}_*^+} \|\mathbf{t}\|^{1/2} \, \mathcal{T}(\mathrm{d}\mathbf{t}) < \infty$. Thus, Proposition 1.1.11 tells us that $\mathbf{Z} \sim FV^{2n}$. Its drift is $\mathbf{m} - \int_{\mathbb{D}_*} (\mathbf{t}, \mathbf{x}) \, \mathcal{Z}_0(\mathrm{d}\mathbf{t}, \mathrm{d}\mathbf{x}) = \mathbf{0}$, where $\mathbf{m}$ is given in (2.2.1)–(2.2.3) and the finiteness of the integral is ensured by $\mathbf{Z} \sim FV^{2n}$. ☐

2.3.7 Moments

In this section, we give formulas for the expected values and covariances of weakly subordinated processes.

Proposition 2.3.22. *Let* $\mathbf{Z} \stackrel{D}{=} (\mathbf{T}, \mathbf{X} \odot \mathbf{T})$. *For* $t > 0$,

$$\frac{\mathbb{E}[\mathbf{T}(t)]}{t} = \mathbf{d} + \int_{[0,\infty)^n_*} \mathbf{t} \, \mathcal{T}(\mathrm{d}\mathbf{t}), \tag{2.3.15}$$

$$\frac{\mathrm{Cov}(\mathbf{T}(t))}{t} = \int_{[0,\infty)^n_*} \mathbf{t}'\mathbf{t} \, \mathcal{T}(\mathrm{d}\mathbf{t}), \tag{2.3.16}$$

$$\begin{aligned}\frac{\mathbb{E}[\mathbf{X} \odot \mathbf{T}(t)]}{t} &= \mathbf{c}(\mathbf{d}, \mathcal{X}) + \mathbf{d} \diamond \boldsymbol{\mu} + \int_{\mathbb{D}^C} \mathbf{x} \, (\mathbf{d} \diamond \mathcal{X})(\mathrm{d}\mathbf{x}) \\ &\quad + \int_{[0,\infty)^n_*} \mathbb{E}[\mathbf{X}(\mathbf{t})] \, \mathcal{T}(\mathrm{d}\mathbf{t}),\end{aligned} \tag{2.3.17}$$

$$\begin{aligned}\frac{\mathrm{Cov}(\mathbf{X} \odot \mathbf{T}(t))}{t} &= \mathbf{d} \diamond \Sigma + \int_{\mathbb{R}^n_*} \mathbf{x}'\mathbf{x} \, (\mathbf{d} \diamond \mathcal{X})(\mathrm{d}\mathbf{x}) \\ &\quad + \int_{[0,\infty)^n_*} \mathbb{E}[\mathbf{X}'(\mathbf{t})\mathbf{X}(\mathbf{t})] \, \mathcal{T}(\mathrm{d}\mathbf{t}),\end{aligned} \tag{2.3.18}$$

$$\frac{\mathrm{Cov}(\mathbf{T}(t), \mathbf{X} \odot \mathbf{T}(t))}{t} = \int_{[0,\infty)^n_*} \mathbf{t}'\mathbb{E}[\mathbf{X}(\mathbf{t})] \, \mathcal{T}(\mathrm{d}\mathbf{t}) \tag{2.3.19}$$

provided the participating integrals are finite.

Proof. Let $t > 0$. We apply Proposition 1.1.9 to the Lévy process $(\mathbf{T}, \mathbf{X} \odot \mathbf{T}) \sim L^{2n}(\mathbf{m}, \Theta, \mathcal{Z})$, where $(\mathbf{m}, \Theta, \mathcal{Z})$ is defined in (2.2.1)–(2.2.5). The first n components of $\mathbb{E}[(\mathbf{T}, \mathbf{X} \odot \mathbf{T})(t))]/t$ give

$$\frac{\mathbb{E}[\mathbf{T}(t)]}{t} = \mathbf{m}_1 + \int_{[0,\infty)^n_*} \mathbf{t}\mathbb{P}((\mathbf{t}, \mathbf{X}(\mathbf{t})) \in \mathbb{D}^C)\, \mathcal{T}(\mathrm{d}\mathbf{t}),$$

from which (2.3.15) follows, and the last n components give

$$\frac{\mathbb{E}[\mathbf{X} \odot \mathbf{T}(t)]}{t} = \mathbf{m}_2 + \int_{\mathbb{D}^C} \mathbf{x}\,(\mathbf{d} \diamond \mathcal{X})(\mathrm{d}\mathbf{x}) + \int_{[0,\infty)^n_*} \mathbb{E}[\mathbf{X}(\mathbf{t})\mathbf{1}_{\mathbb{D}^C}(\mathbf{t}, \mathbf{X}(\mathbf{t}))]\, \mathcal{T}(\mathrm{d}\mathbf{t}),$$

from which (2.3.17) follows.

The covariance matrix of the $2n$-dimensional random vector $(\mathbf{T}(t), \mathbf{X} \odot \mathbf{T}(t))$ satisfies

$$\begin{aligned}\frac{\mathrm{Cov}((\mathbf{T}(t), \mathbf{X} \odot \mathbf{T}(t)))}{t} &= \Theta + \int_{\mathbb{R}^n_*} (\mathbf{0}, \mathbf{x})'(\mathbf{0}, \mathbf{x})\,(\mathbf{d} \diamond \mathcal{X})(\mathrm{d}\mathbf{x}) \\ &\quad + \int_{[0,\infty)^n_*} \mathbb{E}[(\mathbf{t}, \mathbf{X}(\mathbf{t}))'(\mathbf{t}, \mathbf{X}(\mathbf{t}))]\, \mathcal{T}(\mathrm{d}\mathbf{t}).\end{aligned}$$

Upon taking the top-left, bottom-right and top-right $n \times n$ submatrices, we obtain (2.3.16), (2.3.18) and (2.3.19), respectively. □

Remark 2.3.23. Note that (2.3.17) corrects the corresponding formula in Proposition 3.6 of [BLM17], where the term $\mathbf{c}(\mathbf{d}, \mathcal{X})$ is missing.

Example 2.3.24. Consider the case where the subordinate is Brownian motion $\mathbf{B} \sim BM^n(\boldsymbol{\mu}, \Sigma)$ and the subordinator is $\mathbf{T} \sim S^n(\mathbf{d}, \mathcal{T})$. By Proposition 2.3.22 and noting that $\mathbf{c} = \mathbf{0}$, for $1 \le k \le n$, we have

$$\begin{aligned}\mathbb{E}[(\mathbf{B} \odot \mathbf{T})_k(1)] &= d_k\mu_k + \int_{[0,\infty)^n_*} \mu_k t_k\, \mathcal{T}(\mathrm{d}\mathbf{t}) \\ &= \mu_k \mathbb{E}[T_k(1)], \qquad (2.3.20)\\ \mathrm{Var}((\mathbf{B} \odot \mathbf{T})_k(1)) &= d_k\Sigma_{kk} + \int_{[0,\infty)^n_*} (t_k\Sigma_{kk} + \mu_k^2 t_k^2)\, \mathcal{T}(\mathrm{d}\mathbf{t}) \\ &= \Sigma_{kk}\mathbb{E}[T_k(1)] + \mu_k^2\, \mathrm{Var}(T_k(1)). \qquad (2.3.21)\end{aligned}$$

Assume $1 \le k \ne l \le n$, $u > 0$, and let

$$\tau_{k,l}(u) := \mathcal{T}(\{\mathbf{t} = (t_1, \dots, t_n) \in [0,\infty)^n_* : t_k \wedge t_l > u\}).$$

By noting that

$$\int_{[0,\infty)^n_*} t_k \wedge t_l\, \mathcal{T}(\mathrm{d}\mathbf{t}) = \int_{[0,\infty)^n_*} \int_{(0,\infty)} \mathbf{1}_{(u,\infty)}(t_k)\mathbf{1}_{(u,\infty)}(t_l)\, \mathrm{d}u \mathcal{T}(\mathrm{d}\mathbf{t})$$

$$= \int_{(0,\infty)} \tau_{k,l}(u)\,\mathrm{d}u, \tag{2.3.22}$$

Proposition 2.3.22 implies

$$\begin{aligned}
&\mathrm{Cov}((\mathbf{B}\odot\mathbf{T})_k(1), (\mathbf{B}\odot\mathbf{T})_l(1)) \\
&= (d_k \wedge d_l)\Sigma_{kl} + \int_{[0,\infty)^n_*} ((t_k\wedge t_l)\Sigma_{kl} + \mu_k\mu_l t_k t_l)\,\mathcal{T}(\mathrm{d}\mathbf{t}) \\
&= \left(d_k \wedge d_l + \int_{(0,\infty)} \tau_{k,l}(u)\,\mathrm{d}u\right)\Sigma_{kl} + \mu_k\mu_l\,\mathrm{Cov}(T_k(1), T_l(1)).
\end{aligned} \tag{2.3.23}$$

2.3.8 Subordinators with Monotonic Components

Strong subordination may not create a Lévy process when the subordinate $\mathbf{X}$ does not have independent components or when the subordinator $\mathbf{T}$ does not have indistinguishable components, however, it may still be possible that the time marginal distributions of the weakly subordinated process $\mathbf{X}\odot\mathbf{T}(t)$ coincide with that of the strongly subordinated process $\mathbf{X}\circ\mathbf{T}(t)$ for all times $t\geq 0$. In this section, it is shown in Proposition 2.3.26 that this holds if $\mathbf{T}$ has monotonic components, and in Proposition 2.3.29 that the assumption of monotonic components is necessary in some cases.

A closely related question is whether there exists a Lévy process $\mathbf{Y}$, not necessarily a weakly subordinated process, with time marginal distributions $\mathbf{Y}(t)$ matching that of the strongly subordinated process $\mathbf{X}\circ\mathbf{T}(t)$ for all $t\geq 0$. This is partly answered in Proposition 2.3.29, where it is shown that in some cases, such a Lévy process $\mathbf{Y}$ cannot exist.

Definition 2.3.25. An n-dimensional subordinator $\mathbf{T} = (T_1, \dots, T_n)$ has *monotonic components* if there exists a permutation $\langle (1), \dots, (n)\rangle$ such that $T_{(1)} \leq \dots \leq T_{(n)}$.

Proposition 2.3.26. *Let $\mathbf{X}$ and $\mathbf{T}$ be independent. If $\mathbf{T}$ has monotonic components, then $(\mathbf{T}(t), \mathbf{X}\circ\mathbf{T}(t)) \stackrel{D}{=} (\mathbf{T}(t), \mathbf{X}\odot\mathbf{T}(t))$ for all $t\geq 0$.*

Proof. Assume $T_1 \leq \dots \leq T_n$. For $\Sigma = (\Sigma_{ij}) \in \mathbb{R}^{n\times n}$, let $\Sigma_k = (\Sigma_{k,ij}) \in \mathbb{R}^{n\times n}$ be defined by $\Sigma_{k,ij} := \Sigma_{ij}\mathbf{1}_{\{i\wedge j\geq k\}}(i,j)$ for $1\leq k\leq n$, $1\leq i,j\leq n$. Let $\mathbf{t} = (t_1,\dots,t_n) \in [0,\infty)^n_{\leq}$, where

$$[0,\infty)^n_{\leq} := \{\mathbf{t} = (t_1,\dots,t_n) \in [0,\infty)^n : t_1 \leq \dots \leq t_n\},$$

with $\Delta t_k := t_k - t_{k-1}$, $t_0 := 0$ for $1\leq k\leq n$. Introduce the linear bijections $A, D : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{x} = (x_1,\dots,x_n) \in \mathbb{R}^n$,

$$\mathbf{x}A := (x_1, x_1 + x_2, x_1 + x_2 + x_3, \dots, x_1 + x_2 + \dots + x_n),$$
$$\mathbf{x}D := (x_1, x_2 - x_1, x_3 - x_2, \dots, x_n - x_{n-1}).$$

Let $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \mathbb{R}^n$. Let

$$\begin{aligned}
\mathbf{z} &:= -\mathrm{i}\boldsymbol{\theta}_1 A' - \mathrm{i}(\boldsymbol{\theta}_2 \diamond \boldsymbol{\mu})A' + \mathbf{z}_1 - \mathbf{z}_2 - \mathbf{z}_3,\\
\mathbf{z}_1 &:= \frac{1}{2}\sum_{k=1}^n \|\boldsymbol{\theta}_2\|^2_{\Sigma_k}\mathbf{e}_k,\\
\mathbf{z}_2 &:= \sum_{k=1}^n \int_{\mathbb{R}^n_*} (e^{\mathrm{i}\langle\boldsymbol{\theta}_2,\mathbf{x}\rangle} - 1 - \mathrm{i}\langle\boldsymbol{\theta}_2, \mathbf{x}\rangle \mathbf{1}_{\mathbb{D}}(\mathbf{x}))\, \mathcal{X}_{\{k,\dots,n\}}(\mathrm{d}\mathbf{x})\mathbf{e}_k,\\
\mathbf{z}_3 &:= \mathrm{i}\sum_{k=2}^n \int_{\mathbb{D}^C} \langle\boldsymbol{\theta}_2, \mathbf{x}\boldsymbol{\pi}_{\{k,\dots,n\}}\rangle \mathbf{1}_{\mathbb{D}}(\mathbf{x}\boldsymbol{\pi}_{\{k,\dots,n\}})\, \mathcal{X}(\mathrm{d}\mathbf{x})\mathbf{e}_k.
\end{aligned}$$

For each term in $\mathbf{z}$, we now compute its Euclidean product with $\mathbf{t}D$. Firstly, note that $A = D^{-1}$ implies $\langle\boldsymbol{\theta}_1 A', \mathbf{t}D\rangle = \langle\boldsymbol{\theta}_1, \mathbf{t}\rangle$ and $\langle(\boldsymbol{\theta}_2 \diamond \boldsymbol{\mu})A', \mathbf{t}D\rangle = \langle\mathbf{t} \diamond \boldsymbol{\mu}, \boldsymbol{\theta}_2\rangle$. The quantities in (1.2.1)–(1.2.3) become

$$\begin{aligned}
\mathbf{t} \diamond \Sigma &= \sum_{k=1}^n \Delta t_k \Sigma_k,\\
\mathbf{t} \diamond \mathcal{X} &= \sum_{k=1}^n \Delta t_k \mathcal{X}_{\{k,\dots,n\}},\\
\mathbf{c} &= \sum_{k=2}^n \Delta t_k \int_{\mathbb{D}^C} \mathbf{x}\boldsymbol{\pi}_{\{k,\dots,n\}} \mathbf{1}_{\mathbb{D}}(\mathbf{x}\boldsymbol{\pi}_{\{k,\dots,n\}})\, \mathcal{X}(\mathrm{d}\mathbf{x}),
\end{aligned}$$

which respectively imply

$$\begin{aligned}
\langle\mathbf{z}_1, \mathbf{t}D\rangle &= \frac{1}{2}\|\boldsymbol{\theta}_2\|^2_{\mathbf{t}\diamond\Sigma},\\
\langle\mathbf{z}_2, \mathbf{t}D\rangle &= \int_{\mathbb{R}^n_*} (e^{\mathrm{i}\langle\boldsymbol{\theta}_2,\mathbf{x}\rangle} - 1 - \mathrm{i}\langle\boldsymbol{\theta}_2, \mathbf{x}\rangle \mathbf{1}_{\mathbb{D}}(\mathbf{x}))\, (\mathbf{t} \diamond \mathcal{X})(\mathrm{d}\mathbf{x}),\\
\langle\mathbf{z}_3, \mathbf{t}D\rangle &= \mathrm{i}\langle\mathbf{c}, \boldsymbol{\theta}_2\rangle.
\end{aligned}$$

Combining the above results yields

$$-\langle\mathbf{z}, \mathbf{t}D\rangle = \mathrm{i}\langle\boldsymbol{\theta}_1, \mathbf{t}\rangle + (\mathbf{t} \diamond \Psi_{\mathbf{X}})(\boldsymbol{\theta}_2). \tag{2.3.24}$$

Since $\mathbf{T}$ and $\mathbf{X}$ are independent and $\mathbf{T}$ is supported on $[0,\infty)^n_{\leq}$, (2.3.24) implies $\Phi_{(\mathbf{T}(t),\mathbf{X}\circ\mathbf{T}(t))}(\boldsymbol{\theta}) = \mathbb{E}[\exp(-\langle\mathbf{z}, \mathbf{T}D(t)\rangle)]$. Applying Theorem 24.11 in [Sat99], each component of $\mathbf{T}D$ is a subordinator since its support is bounded below by 0 due to the assumption $T_1 \le \ldots \le T_n$. So by Proposition 1.1.8, $\mathbf{T}D \sim S^n(\mathbf{d}D, \mathcal{T} \circ D^{-1})$. By noting that $\Re\mathbf{z} \in [0,\infty)^n$ and applying Proposition 1.1.14, we have $\Phi_{(\mathbf{T}(t),\mathbf{X}\circ\mathbf{T}(t))}(\boldsymbol{\theta}) = \exp(-t\Lambda_{\mathbf{T}D}(\mathbf{z}))$, where

$$\Lambda_{\mathbf{T}D}(\mathbf{z}) = \langle \mathbf{d}D, \mathbf{z}\rangle + \int_{[0,\infty)^n_*} (1 - e^{-\langle \mathbf{z}, \mathbf{t}D\rangle})\, \mathcal{T}(\mathrm{d}\mathbf{t})$$

due to the transformation theorem (see Proposition A.3.1) and $\mathcal{T}([0,\infty)^n_* \backslash [0,\infty)^n_{\le}) = 0$. By noting $\mathbf{d} \in [0,\infty)^n_{\le}$, (2.3.24) and $e^{-\langle \mathbf{z}, \mathbf{t}D\rangle} = \Phi_{(\mathbf{t},\mathbf{X}(\mathbf{t}))}(\boldsymbol{\theta})$ for $\mathbf{t} \in [0,\infty)^n_{\le}$, we have that $-\Lambda_{\mathbf{T}D}(\mathbf{z})$ matches the RHS of (2.3.1). Thus, $\Phi_{(\mathbf{T}(t),\mathbf{X}\circ\mathbf{T}(t))}(\boldsymbol{\theta}) = \Phi_{(\mathbf{T}(t),\mathbf{X}\odot\mathbf{T}(t))}(\boldsymbol{\theta})$, so $(\mathbf{T}(t), \mathbf{X}\circ\mathbf{T}(t)) \overset{D}{=} (\mathbf{T}(t), \mathbf{X}\odot\mathbf{T}(t))$ for all $t \ge 0$.

Now assume, instead of $T_1 \le \cdots \le T_n$, that there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that $\mathbf{T}P$ satisfies $(\mathbf{T}P)_1 \le \cdots \le (\mathbf{T}P)_n$. Since $\mathbf{T}P$ is an n-dimensional subordinator, we have proved $(\mathbf{T}P(t), \mathbf{X}P \circ \mathbf{T}P(t)) \overset{D}{=} (\mathbf{T}P(t), \mathbf{X}P \odot \mathbf{T}P(t))$ for $t \ge 0$, which implies

$$(\mathbf{T}(t), \mathbf{X}\circ\mathbf{T}(t)) = (\mathbf{T}(t), (\mathbf{X}P \circ \mathbf{T}P)P^{-1}(t)) \overset{D}{=} (\mathbf{T}(t), (\mathbf{X}P \odot \mathbf{T}P)P^{-1}(t))$$

for $t \ge 0$. Then applying Corollary 2.3.8 to the subordinator $\mathbf{T}P$, the Lévy process $\mathbf{X}P$ and the permutation matrix P^{-1} yields $(\mathbf{T}(t), \mathbf{X}\circ\mathbf{T}(t)) \overset{D}{=} (\mathbf{T}(t), \mathbf{X}\odot\mathbf{T}(t))$ for $t \ge 0$, which completes the . □

Example 2.3.27. Recall Example 2.2.6. The deterministic subordinator $(I, 2I)$ satisfies $I \le 2I$. By Proposition 2.3.26, $\mathbf{Z}(t) \overset{D}{=} (t, 2t, B(t), B(2t))$ for all $t \ge 0$.

Example 2.3.28. Let B, B^*, N be independent processes, where B, B^* are standard Brownian motions and N is a Poisson process with unit rate. The subordinator (I, N) does not have monotonic components, and $(B, B)\circ(I, N)$ is not a Lévy process because, by conditioning on N,

$$\mathbb{E}[B(t)B(N(t))] = \mathbb{E}[t \wedge N(t)] = t(1 - e^{-t}), \quad 0 \le t \le 1,$$

is a nonlinear function in t, contradicting Proposition 1.1.9. So there can be no Lévy process matching $(B, B)\circ(I, N)$ in law at all times $t \ge 0$. In contrast,

$$((I, N), (B, B) \odot (I, N)) \overset{D}{=} (I, N, B, B^* \circ N)$$

using (2.2.1)–(2.2.5).

Let $\mathbf{X}$ and $\mathbf{T}$ be independent. Example 2.3.28 shows that without the assumption of monotonic components in Proposition 2.3.26, it is possible that the conclusion

$\mathbf{X} \circ \mathbf{T}(t) \stackrel{D}{=} \mathbf{X} \odot \mathbf{T}(t)$ for all $t \geq 0$ is violated, and there may be no Lévy process whose time marginal distributions match that of $\mathbf{X} \circ \mathbf{T}(t)$ for all $t \geq 0$. This is developed further in the next proposition, showing the necessity of the monotonic component assumption in some cases.

Proposition 2.3.29. *Suppose $n \geq 2$. Let $\mathbf{T}$ and $\mathbf{X}$ be n-dimensional Lévy processes, where $\mathbf{T}$ and $\mathbf{X}$ are independent, $\mathbf{T}$ is a subordinator and $\mathbf{X}$ has dependent components. Assume that all components of $\mathbf{T}$ are nonzero. If $\mathbf{Y}$ is an n-dimensional Lévy process, then there exists $t > 0$ violating $\mathbf{Y}(t) \stackrel{D}{=} \mathbf{X} \circ \mathbf{T}(t)$ provided that one of the following holds:*

(i) both $\mathbf{T}$ and $\mathbf{X}$ admit finite second moments, $\mathbf{X}$ has correlated components and $\mathbf{T}$ has non-monotonic components;

(ii) $\mathbf{Y} \stackrel{D}{=} \mathbf{X} \odot \mathbf{T}$, $\mathbf{X} \stackrel{D}{=} -\mathbf{X}$ is symmetric and $\mathbf{T}$ is driftless with independent components.

Proof. Let $\mathbf{T} = (T_1, \dots, T_n)$, $\mathbf{X} = (X_1, \dots, X_n)$ and $\mathbf{Y} = (Y_1, \dots, Y_n)$. For Part (i), let $1 \leq k \neq l \leq n$ be the index of any two components of $\mathbf{X}$ that are correlated and note that proving the proposition with $\mathbf{T}$, $\mathbf{X}$, $\mathbf{Y}$ replaced by (T_k, T_l), (X_k, X_l), (Y_k, Y_l), respectively, is sufficient. This reasoning also holds for Part (ii) by letting $1 \leq k \neq l \leq n$ be the index of any two components of $\mathbf{X}$ that are dependent and recalling Corollary 2.3.7. Thus, for the remainder of the proof, we assume without loss of generality that $n = 2$.

Let the bivariate subordinator $\mathbf{T} = (T_1, T_2)$ and the bivariate Lévy process $\mathbf{X} = (X_1, X_2)$ be independent. Let $D := T_2 - T_1 \sim FV^1(d, \mathcal{D})$. Let $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}^2$ and $r, s \geq 0$. Recall the notation $x^+ = x \vee 0$ and $x^- = -(x \wedge 0)$, $x \in \mathbb{R}$. Using (1.2.6), we have

$$(r, s) \diamond \Psi_{\mathbf{X}}(\boldsymbol{\theta}) = (r \wedge s)\Psi_{\mathbf{X}}(\boldsymbol{\theta}) + (s - r)^+ \Psi_{X_2}(\theta_2) + (s - r)^- \Psi_{X_1}(\theta_1),$$

and thus, by conditioning on $\mathbf{T}$ and using Proposition 1.2.2, we have

$$\begin{aligned}\Phi_{\mathbf{X}\circ\mathbf{T}(t)}(\boldsymbol{\theta}) = \mathbb{E}[\exp((T_1(t) \wedge T_2(t))\Psi_{\mathbf{X}}(\boldsymbol{\theta}) + D^+(t)\Psi_{X_2}(\theta_2) \\ + D^-(t)\Psi_{X_1}(\theta_1))].\end{aligned} \tag{2.3.25}$$

(i). By Definition 2.3.25, $\mathbf{T}$ has monotonic components if and only if D or $-D$ is a subordinator. As we assumed $\mathbf{T}$ to have non-monotonic and non-deterministic components, one of the following exclusive cases holds (see Corollary 24.8 and Theorem 24.10 in [Sat99]):

(a) $\mathcal{D}((-\infty, 0)) > 0$, $\mathcal{D}((0, \infty)) = 0$ and $d > 0$, so that the support of $D(1)$ is unbounded towards $-\infty$ with d as its supremum;

(b) $\mathcal{D}((-\infty, 0)) = 0$, $\mathcal{D}((0, \infty)) > 0$ and $d < 0$, so that the support of $D(1)$ is unbounded towards ∞ with d as its infimum;

(c) $\mathcal{D}((-\infty, 0)) > 0$, $\mathcal{D}((0, \infty)) > 0$ and $d \in \mathbb{R}$, so that the support of $D(1)$ is unbounded towards ∞ and $-\infty$.

In all cases, we have $\mathbb{P}(D(1) > 0) > 0$ and $\mathbb{P}(D(1) < 0) > 0$, implying $\mathbb{E}[D^+(1)] > 0$ and $\mathbb{E}[D^-(1)] > 0$, respectively.

Assume for the purpose of contradiction that $\mathbb{E}[D^+(t)] = t\mathbb{E}[D^+(1)]$ for all $t \geq 0$ so that $\mathbb{E}[D^+(1)] = \mathbb{E}[(D(n)/n)^+]$, $n \in \mathbb{N}$. By assumption, $\mathbf{T}$ has a second moment, so D does too, implying $\mathbb{E}[|D(1)|] = \mathbb{E}[D^+(1)] + \mathbb{E}[D^-(1)] < \infty$. So as $n \to \infty$,

$$E_n := \frac{D(n)}{n} = \frac{1}{n}\sum_{k=1}^{n}(D(k) - D(k-1)) \xrightarrow{\text{a.s.}} \mathbb{E}[D(1)].$$

by the strong law of large numbers. Then by the dominated convergence theorem, this also holds in mean, giving $\mathbb{E}[|E_n - a|] \to 0$ as $n \to \infty$, where $a := \mathbb{E}[D(1)]$. Let $f(x) := x^+$, $x \in \mathbb{R}$, then $\mathbb{E}[|f(E_n) - f(a)|] \leq \mathbb{E}[K|E_n - a|] \to 0$ as $n \to \infty$, where $K = 1$ is the Lipschitz constant of f. Consequently, we have

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\frac{D(n)}{n}\right)^+\right] = \mathbb{E}[\mathbb{E}[D(1)]^+] = \mathbb{E}[D(1)]^+.$$

This leads to the contradiction

$$\mathbb{E}[D^+(1)] = \mathbb{E}[D(1)]^+ = (\mathbb{E}[D^+(1)] - \mathbb{E}[D^-(1)])^+ < \mathbb{E}[D^+(1)].$$

To summarise, $t \mapsto \mathbb{E}[D^+(t)]$, $t \geq 0$, cannot be a linear function.

On the RHS of (2.3.25), taking partial derivatives twice with respect to $\boldsymbol{\theta}$ under the expectation and applying dominated convergence when $\boldsymbol{\theta} \to \mathbf{0}$, we get

$$\begin{aligned}\operatorname{Cov}(X_1 \circ T_1(t), X_2 \circ T_2(t)) = \mathbb{E}[X_1(1)]\mathbb{E}[X_2(1)] \operatorname{Cov}(T_1(t), T_2(t)) \\ + \rho\mathbb{E}[T_1(t) \wedge T_2(t)],\end{aligned} \tag{2.3.26}$$

where $\rho := \operatorname{Cov}(X_1(1), X_2(1))$. By our assumptions, $\mathbf{T}$ and $\mathbf{X}$ admit finite second moments, so that both sides of (2.3.26) are finite.

For purpose of contradiction, assume that $\mathbf{Y}(t) \stackrel{D}{=} \mathbf{X} \circ \mathbf{T}(t)$ for all $t \geq 0$, where $\mathbf{Y}$ is a bivariate Lévy process. Thus, $\mathbf{T}$ and $\mathbf{Y}$ are Lévy processes with finite second moments. In particular, $t \mapsto \mathrm{Cov}(T_1(t), T_2(t))$ and $t \mapsto \mathrm{Cov}(Y_1(t), Y_2(t))$ are linear functions, and so is $t \mapsto \mathbb{E}[T_1(t) \wedge T_2(t)]$ as we assume $\rho \neq 0$ in (2.3.26).

Also, $t \mapsto \mathbb{E}[T_2(t)]$ is linear, so noting that $\mathbb{E}[T_1(t) \wedge T_2(t)] = \mathbb{E}[T_2(t)] - \mathbb{E}[D^+(t)]$, $t \geq 0$, contradicts the non-linearity of $t \mapsto \mathbb{E}[D^+(t)]$, completing the proof of Part (i).

(ii). If T_1, T_2 are independent and driftless, the components of $\mathbf{X} \odot \mathbf{T}$ are independent by Proposition 2.3.18. Then using Theorem 2.3.5 on each component yields $\mathbf{X} \odot \mathbf{T} \stackrel{D}{=} (X_1 \circ T_1, X_2^* \circ T_2)$ for independent Lévy processes T_1, T_2, X_1, X_2^*, where $X_2^* \stackrel{D}{=} X_2$.

Let $\boldsymbol{\theta} \in \mathbb{R}^2$. By conditioning on $\mathbf{T}$ and noting that $X_1 \circ T_1$ and $X_2^* \circ T_2$ are independent, we have

$$\Phi_{\mathbf{X} \odot \mathbf{T}(t)}(\boldsymbol{\theta}) = \mathbb{E}[\exp(T_1(t)\Psi_{X_1}(\theta_1) + T_2(t)\Psi_{X_2}(\theta_2))]. \tag{2.3.27}$$

Next, recall the definition of $\widehat{\Psi}_{\mathbf{X}}$ in (1.3.6). For $r, s \geq 0$, $z, z_1, z_2 \in \mathbb{C}$, $\widehat{z} := z - z_1 - z_2$, note that

$$(r \wedge s)z + (s-r)^+ z_2 + (s-r)^- z_1 = (r \wedge s)\widehat{z} + r z_1 + s z_2,$$

so that (2.3.25) becomes

$$\begin{aligned}\Phi_{\mathbf{X} \circ \mathbf{T}(t)}(\boldsymbol{\theta}) = \mathbb{E}[\exp((T_1(t) \wedge T_2(t))\widehat{\Psi}_{\mathbf{X}}(\boldsymbol{\theta}) + T_1(t)\Psi_{X_1}(\theta_1) \\ + T_2(t)\Psi_{X_2}(\theta_2))].\end{aligned} \tag{2.3.28}$$

Since X_1 and X_2 are dependent, there exists $\boldsymbol{\theta} \in \mathbb{R}^2$ such that $\widehat{\Psi}_{\mathbf{X}}(\boldsymbol{\theta}) \neq 0$. Further, the symmetry assumption $\mathbf{X} \stackrel{D}{=} -\mathbf{X}$ implies $\Psi_{X_k}(\theta_k), \widehat{\Psi}_{\mathbf{X}}(\boldsymbol{\theta}) \in \mathbb{R}$, $k = 1, 2$. If (2.3.27) matches (2.3.28) for all $t > 0$, we have

$$\mathbb{E}[\exp(T_1(t)\Psi_{X_1}(\theta_1) + T_2(t)\Psi_{X_2}(\theta_2))(\exp((T_1(t) \wedge T_2(t))\widehat{\Psi}_{\mathbf{X}}(\boldsymbol{\theta})) - 1)] = 0,$$

which implies $T_1(t) \wedge T_2(t) = 0$ a.s. for all $t > 0$. In particular, $T_1 \wedge T_2$ must be a zero process, which contradicts T_1, T_2 being nonzero, completing the proof. □

Remark 2.3.30. In Proposition 2.3.29, when assuming (i), the conclusion that there is no Lévy process $\mathbf{Y}$ such that $\mathbf{Y}(t) \stackrel{D}{=} \mathbf{X} \circ \mathbf{T}(t)$ for all $t \geq 0$ is more general than $\mathbf{X} \odot \mathbf{T}(t) \stackrel{D}{=} \mathbf{X} \circ \mathbf{T}(t)$ failing to hold for some time $t > 0$.

Under the assumptions in (ii), $\mathbf{T} = (T_1, \dots, T_n) \sim S^n(\mathbf{0}, \mathcal{T})$ has non-monotonic components. Suppose the contrary for the purpose of contradiction, so there exists a permutation matrix $P \in \mathbb{R}^{n \times n}$ such that $(\mathbf{T}P)_1 \leq \dots \leq (\mathbf{T}P)_n$. Introduce the linear transformation $C : [0, \infty)^n \to \mathbb{R}^{n-1}$, $\mathbf{t} = (t_1, \dots, t_n) \in [0, \infty)^n$, $\mathbf{t}C :=$

$(t_2 - t_1, t_3 - t_2, \dots, t_n - t_{n-1})$. Then $\mathbf{S} := \mathbf{T}PC \sim S^{n-1}(\mathbf{0}, \mathcal{S})$ must be a subordinator. Its Lévy measure is $\mathcal{S} = \mathcal{T} \circ (PC)^{-1}$ due to Proposition 1.1.8. Let $\langle (1), \dots (n) \rangle$ be the permutation associated with P and $\mathcal{T}_{(1)}$ be the Lévy measure of $T_{(1)}$. We have

$$\mathcal{S}((-\infty, 0]^{n-1}) = \mathcal{T}(\{\mathbf{t} \in \mathbb{R}^n : t_{(1)} \leq \cdots \leq t_{(n)}\}) \geq \mathcal{T}_{(1)}((0, \infty)) > 0,$$

where the first inequality follows from (2.3.14) by noting that $\mathbf{T}$ has independent components, and the second inequality follows from $T_{(1)}$ being driftless and nonzero. Therefore, the support of $\mathcal{S}$ extends outside of $[0, \infty)_*^{n-1}$ so that, by Proposition 1.1.14, $\mathbf{S}$ cannot be a subordinator, which is a contradiction.

Chapter 3

Weak Variance Generalised Gamma Convolutions

Our first major application of weak subordination is to construct the multivariate class of weak variance generalised gamma convolutions. These are processes formed by weakly subordinating Brownian motions and Thorin subordinators and unifies the processes in Grigelionis [Gri07b] and Buchmann et al. [BKMS17] formed by univariate and multivariate subordination, respectively.

In Section 3.1, we recall definitions and properties of Thorin subordinators and derive a representation for their Lévy measure. In Section 3.2, we introduce weak variance generalised gamma convolutions and derive their characteristic triplet, characteristic function and Lévy density. In Section 3.3, we provide a condition for the q-variation of these processes to be finite in terms of that of their subordinator or a moment condition on their Thorin measure, and we also discuss finite variation.

3.1 Thorin Subordinators

The generalised gamma convolution class on the cone $[0,\infty)^n$, denoted GGC^n, is the minimal class of random vectors of the form $G\boldsymbol{\alpha}$, where G is a gamma random variable, $\boldsymbol{\alpha} \in [0,\infty)^n$, while being closed under convolution and convergence in distribution [PAS14]. Since GGC^n distributions are infinitely divisible, their associated Lévy processes exist and are known as Thorin subordinators. This is a rich class of subordinators which we use to construct the weak variance generalised gamma convolutions. Following the exposition in [BKMS17], this section provides a review of Thorin subordinators. Recall that $\ln^- x = (\ln x)^-$, $x > 0$.

Definition 3.1.1. An n-dimensional nonnegative Borel measure $\mathcal{U}$ on $[0,\infty)^n_*$ satisfying

$$\int_{[0,\infty)^n_*} (1 + \ln^- \|\mathbf{u}\|) \wedge (\|\mathbf{u}\|^{-1}) \mathcal{U}(\mathrm{d}\mathbf{u}) < \infty \tag{3.1.1}$$

is a *Thorin measure.*

Definition 3.1.2. Let $\mathbf{d} \in [0,\infty)^n$ and $\mathcal{U}$ be an n-dimensional Thorin measure. An n-dimensional subordinator $\mathbf{T} \sim GGC^n_S(\mathbf{d},\mathcal{U})$ is a *Thorin subordinator* if it has Laplace exponent

$$\Lambda_{\mathbf{T}}(\boldsymbol{\lambda}) = \langle \mathbf{d}, \boldsymbol{\lambda} \rangle + \int_{[0,\infty)^n_*} \ln\left(1 + \frac{\langle \boldsymbol{\lambda}, \mathbf{u} \rangle}{\|\mathbf{u}\|^2}\right) \mathcal{U}(\mathrm{d}\mathbf{u}), \quad \boldsymbol{\lambda} \in [0,\infty)^n. \tag{3.1.2}$$

From this definition, it is clear that the law of a Thorin subordinator is characterised by the parameters $\mathbf{d}$ and $\mathcal{U}$.

Example 3.1.3. If $G \sim \Gamma_S(a,b)$ is a gamma subordinator, then it is a Thorin subordinator because $G \sim GGC^1_S(0, a\boldsymbol{\delta}_b)$.

In the lemma below, we give the polar representation of a Thorin measure and the corresponding Lévy measure. Recall that $\mathbb{S} := \{\mathbf{s} \in \mathbb{R}^n : \|\mathbf{s}\| = 1\}$ and $\mathbb{S}_+ = \mathbb{S} \cap [0,\infty)^n$.

Lemma 3.1.4. *Let $\mathbf{T} \sim GGC^n_S(\mathbf{d},\mathcal{U})$ be a Thorin subordinator with Lévy measure $\mathcal{T}$. Let $A \subseteq [0,\infty)^n_*$ be a Borel set. There exists a finite Borel measure $\mathcal{S}$ on $\mathbb{S}_+$ and a Borel transition kernel $\mathcal{K}$ from $\mathbb{S}_+$ to $(0,\infty)$ with*

$$0 < \int_{(0,\infty)} (1 + \ln^- r) \wedge r^{-1} \mathcal{K}(\mathbf{s}, \mathrm{d}r) < \infty$$

for all $\mathbf{s} \in \mathbb{S}_+$ such that the Thorin measure satisfies

$$\mathcal{U}(A) = (\mathcal{S} \otimes \mathcal{K}) \circ ((\mathbf{s}, r) \mapsto r\mathbf{s})^{-1}(A) = \int_{\mathbb{S}_+} \int_{(0,\infty)} \mathbf{1}_A(r\mathbf{s}) \mathcal{K}(\mathbf{s}, \mathrm{d}r) \mathcal{S}(\mathrm{d}\mathbf{s}). \tag{3.1.3}$$

In addition, the Lévy measure satisfies

$$\mathcal{T}(A) = \int_{\mathbb{S}+} \int_{(0,\infty)} \mathbf{1}_A(r\mathbf{s}) k(\mathbf{s}, r) \frac{\mathrm{d}r}{r} \mathcal{S}(\mathrm{d}\mathbf{s}), \tag{3.1.4}$$

$$k(\mathbf{s}, r) := \int_{(0,\infty)} e^{-rv} \mathcal{K}(\mathbf{s}, \mathrm{d}v), \quad r > 0, \quad \mathbf{s} \in \mathbb{S}_+. \tag{3.1.5}$$

Proof. For (3.1.3), see Lemma 4.1 in [BKMS17]. For (3.1.4) and (3.1.5), see Equations (2.17) and (2.18) in [BKMS17]. □

Remark 3.1.5. Note that the measures $\mathrm{d}\mathbf{x}$, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{U}$ are σ-finite, and the transition kernel $\mathcal{K}$ is locally finite relative to $(0,\infty)$ (see Lemma 4.1 in [BKMS17]) so, in particular, it is σ-finite. Therefore, we can freely interchange integrals involving these measures using Fubini's theorem.

The next lemma gives the characteristics of a Thorin subordinator, expressing its Lévy measure in terms of $\mathcal{G}_b$, the Lévy measure of a standard gamma subordinator with shape parameter b, given in (1.1.8).

Lemma 3.1.6. *If* $\mathbf{T} \sim GGC^n_S(\mathbf{d},\mathcal{U})$*, then* $\mathbf{T} \sim S^n(\mathbf{d},\mathcal{T})$*, where*

$$\mathcal{T} = \left(\frac{\mathcal{U}(\mathrm{d}\mathbf{u})}{\|\mathbf{u}\|^2} \otimes \mathcal{G}_{\|\mathbf{u}\|^2}(\mathrm{d}g)\right) \circ ((\mathbf{u},g) \mapsto g\mathbf{u})^{-1}. \tag{3.1.6}$$

Proof. Let $A \subseteq [0,\infty)^n_*$ be a Borel set. Evaluating the RHS of (3.1.6) at A gives

$$\begin{aligned}\int_{[0,\infty)^n_*}\int_{(0,\infty)} \mathbf{1}_A(g\mathbf{u})e^{-\|\mathbf{u}\|^2 g}\frac{\mathrm{d}g}{g}\mathcal{U}(\mathrm{d}\mathbf{u}) &= \int_{[0,\infty)^n_*}\int_{(0,\infty)} \mathbf{1}_A\left(\frac{r\mathbf{u}}{\|\mathbf{u}\|}\right)e^{-\|\mathbf{u}\|r}\frac{\mathrm{d}r}{r}\mathcal{U}(\mathrm{d}\mathbf{u}) \\ &= \int_{\mathbb{S}^+}\int_{(0,\infty)}\left(\int_{(0,\infty)} \mathbf{1}_A(r\mathbf{s})e^{-rv}\frac{\mathrm{d}r}{r}\right)\mathcal{K}(\mathbf{s},\mathrm{d}r)\mathcal{S}(\mathrm{d}\mathbf{s}) \\ &= \mathcal{T}(A),\end{aligned}$$

where the first equality follows by making the substitution $g = r/\|\mathbf{u}\|$, the second equality follows from using the polar representation (3.1.3) to evaluate the integral with respect to $\mathcal{U}$, and the third equality follows from (3.1.4) and Remark 3.1.5. This completes the proof. □

3.2 Characteristics

Weak variance generalised gamma convolutions are constructed by taking the weak subordination of a Brownian motion $\mathbf{B}$ and a Thorin subordinator $\mathbf{T}$. In this section, we derive its characteristics noting that we obtain simplifications due to the Lévy measure of the Brownian motion being 0.

From now on, we take the weakly subordinated process to be of the form $\mathbf{Y} \stackrel{D}{=} \mathbf{B} \odot \mathbf{T}$, unless otherwise stated.

Definition 3.2.1. Let $\mathbf{d} \in [0,\infty)^n$, $\boldsymbol{\mu} \in \mathbb{R}^n$, $\Sigma \in \mathbb{R}^{n\times n}$ be a covariance matrix and $\mathcal{U}$ be an n-dimensional Thorin measure. An n-dimensional Lévy process $\mathbf{Y} \sim VGG^n(\mathbf{d},\boldsymbol{\mu},\Sigma,\mathcal{U})$ is a *weak variance generalised gamma convolution* if $\mathbf{Y} \stackrel{D}{=} \mathbf{B} \odot \mathbf{T}$, where $\mathbf{B} \sim BM^n(\boldsymbol{\mu},\Sigma)$ and $\mathbf{T} \sim GGC^n_S(\mathbf{d},\mathcal{U})$.

Theorem 2.2.4 ensures the existence of $\mathbf{Y} \sim VGG^n(\mathbf{d}, \boldsymbol{\mu}, \Sigma, \mathcal{U})$.

Definition 3.2.2. Let $d \geq 0$, $\boldsymbol{\mu} \in \mathbb{R}^n$, $\Sigma \in \mathbb{R}^{n\times n}$ be a covariance matrix and $\mathcal{U}$ be a univariate Thorin measure. An n-dimensional Lévy process $\mathbf{Y} \sim VGG^{n,1}(d, \boldsymbol{\mu}, \Sigma, \mathcal{U})$ is a *variance univariate generalised gamma convolution* if $\mathbf{Y} \stackrel{D}{=} \mathbf{B} \circ (T\mathbf{e})$, where $\mathbf{B} \sim BM^n(\boldsymbol{\mu}, \Sigma)$ and $T \sim GGC^1_S(d, \mathcal{U})$ are independent.

Definition 3.2.3. Let $\mathbf{d} \in [0, \infty)^n$, $\boldsymbol{\mu} \in \mathbb{R}^n$, $\Sigma \in \mathbb{R}^{n\times n}$ be a diagonal covariance matrix and $\mathcal{U}$ be an n-dimensional Thorin measure. An n-dimensional Lévy process $\mathbf{Y} \sim VGG^{n,n}(\mathbf{d}, \boldsymbol{\mu}, \Sigma, \mathcal{U})$ is a *variance multivariate generalised gamma convolution* if $\mathbf{Y} \stackrel{D}{=} \mathbf{B} \circ \mathbf{T}$, where $\mathbf{B} \sim BM^n(\boldsymbol{\mu}, \Sigma)$ and $\mathbf{T} \sim GGC^n_S(\mathbf{d}, \mathcal{U})$ are independent.

The $VGG^{n,1}$ class was introduced in [Gri07b], though the name came from [BKMS17], and the $VGG^{n,n}$ class was introduced in [BKMS17]. As a result of Theorem 2.3.5, we have $VGG^n \supseteq VGG^{n,1} \cup VGG^{n,n}$, and the next example explicitly gives the parametrisation.

Example 3.2.4. We have $\mathbf{Y} \sim VGG^{n,1}(d, \boldsymbol{\mu}, \Sigma, \mathcal{U}_0)$ if and only if $\mathbf{Y} \sim VGG^n(d\mathbf{e}, \boldsymbol{\mu}, \Sigma, \mathcal{U})$, where

$$\mathcal{U} = \int_{(0,\infty)} \boldsymbol{\delta}_{u\mathbf{e}} \mathcal{U}_0(\mathrm{d}u).$$

By Example 3.1.3, $\mathbf{V} \sim VG^n(b, \boldsymbol{\mu}, \Sigma)$ if and only if $\mathbf{V} \sim VGG^{n,1}(0, \boldsymbol{\mu}, \Sigma, b\boldsymbol{\delta}_b)$. Also, $\mathbf{Y} \sim VGG^{n,n}(\mathbf{d}, \boldsymbol{\mu}, \Sigma, \mathcal{U})$ if and only if $\mathbf{Y} \sim VGG^n(\mathbf{d}, \boldsymbol{\mu}, \Sigma, \mathcal{U})$ and Σ is diagonal.

Remark 3.2.5. In analogy to Theorem 2.5 in [BKMS17], for $\emptyset \neq J \subseteq \{1, \ldots, n\}$, we introduce

$$\begin{aligned} C_J &:= \{\mathbf{u} = (u_1, \ldots, u_n) \in [0,\infty)^n_* : u_j > 0 \text{ if } j \in J,\, u_j = 0 \text{ if } j \notin J\}, \\ V_J &:= \{\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n_* : x_j \neq 0 \text{ if } j \in J,\, x_j = 0 \text{ if } j \notin J\}, \end{aligned}$$

so that we can partition the spaces $[0,\infty)^n_* = \bigcup_J C_J$ and $\mathbb{R}^n_* = \bigcup_J V_J$. Sums and unions indexed by J are taken over $\emptyset \neq J \subseteq \{1, \ldots, n\}$.

Any Thorin subordinator $\mathbf{T} \sim GGC^n_S(\mathbf{d}, \mathcal{U})$ can be written as a superposition of $\mathbf{d}I$ and independent driftless Thorin subordinators $\mathbf{T}_J$, $\emptyset \neq J \subseteq \{1, \ldots, n\}$, by letting

$$\mathbf{T} \stackrel{D}{=} \mathbf{d}I + \sum_J \mathbf{T}_J, \quad \mathbf{T}_J \sim GGC^n_S(\mathbf{0}, \mathcal{U}_J), \quad \mathcal{U}_J(\mathrm{d}\mathbf{u}) := 1_{C_J}(\mathbf{u})\mathcal{U}(\mathrm{d}\mathbf{u}).$$

This follows from the Laplace exponent in (3.1.2). Note that by (3.1.6), $\mathbf{T}_J$ has Lévy measure $\mathcal{T}_J(\mathrm{d}\mathbf{t}) := 1_{C_J}(\mathbf{t})\mathcal{T}(\mathrm{d}\mathbf{t})$, where $\mathcal{T}$ is the Lévy measure of $\mathbf{T}$. So we obtain the interpretation $\mathbf{T}_J(t) = \sum_{s\in(0,t]} \mathbf{1}_{C_J}(\Delta\mathbf{T}(s))\Delta\mathbf{T}(s)$, $t \geq 0$. Let $\mathbf{B} \sim BM^n(\boldsymbol{\mu}, \Sigma)$. Then by Proposition 2.3.15, $\mathbf{Y} \sim VGG^n(\mathbf{d}, \boldsymbol{\mu}, \Sigma, \mathcal{U})$ can be written as the superposition of independent VGG^n processes $\mathbf{Y}_J$, $J \subseteq \{1, \ldots, n\}$, with

$$\mathbf{Y} \stackrel{D}{=} \mathbf{Y}_\emptyset + \sum_J \mathbf{Y}_J, \quad \mathbf{Y}_\emptyset \stackrel{D}{=} \mathbf{B} \odot (\mathbf{d}I), \quad \mathbf{Y}_J \stackrel{D}{=} \mathbf{B} \odot \mathbf{T}_J.$$

Here, $\mathbf{Y}_J$ is supported on at most a $\#J$-dimensional subspace of $\mathbb{R}^n$ and has Lévy measure $\mathcal{Y}_J$.

Let $\mathbf{u} = (u_1, \ldots, u_n) \in (0, \infty)^n$ and recall the notation $\prod \mathbf{u} := \prod_{k=1}^n u_k$. Let Σ be an invertible covariance matrix. Then $\mathbf{u} \diamond \Sigma$ is a covariance matrix by Lemma 1.2.1, and it is also invertible because

$$|\mathbf{u} \diamond \Sigma| \geq \Big(\prod \mathbf{u}\Big)|\Sigma| > 0 \tag{3.2.1}$$

due to Lemma A.2.1. If $\mathbf{u} \in C_J$ and Σ is invertible, the restriction $(\mathbf{u} \diamond \Sigma)_J : \mathbb{R}^n\boldsymbol{\pi}_J \to \mathbb{R}^n\boldsymbol{\pi}_J$, $\mathbf{x} \mapsto \mathbf{x}(\mathbf{u} \diamond \Sigma)_J := \mathbf{x}(\mathbf{u} \diamond \Sigma)$ is an invertible linear transformation by a $\#J$-dimensional application of the result in (3.2.1). Thus, $(\mathbf{u} \diamond \Sigma)_J$ has an inverse denoted $(\mathbf{u} \diamond \Sigma)_J^{-1}$ and a determinant denoted $|\mathbf{u} \diamond \Sigma|_J > 0$. In matrix notation, $(\mathbf{u} \diamond \Sigma)_J^{-1}$ is the $n \times n$ matrix that is 0 everywhere, except the submatrix formed by keeping only the rows and columns in the index set J is the inverse of the submatrix of $\mathbf{u} \diamond \Sigma$ formed by keeping only the rows and columns in the index set J.

We will see in the proof of Theorem 3.2.6 that the above decomposition is required so that we can apply the VG^n Lévy density formula (1.3.5) for $\mathbf{u} \in [0, \infty)^n_* \backslash (0, \infty)^n$, where $\mathbf{u} \diamond \Sigma$ does not satisfy the invertibility condition but $(\mathbf{u} \diamond \Sigma)_J$ does.

Recall that $\mathcal{V}_{b,\boldsymbol{\mu},\Sigma}$, the Lévy measure of a VG^n process, is given in (1.3.3), and $\mathfrak{K}_\rho$ is defined in (1.3.2). For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_*$, let

$$\mathcal{L}_J(\mathrm{d}\mathbf{x}) := \bigotimes_{k=1}^n (\mathbf{1}_J(k)\mathrm{d}x_k + \mathbf{1}_{\{1,\ldots,n\}\backslash J}(k)\boldsymbol{\delta}_0(\mathrm{d}x_k)).$$

Theorem 3.2.6. *Let* $\mathbf{d} \in [0, \infty)^n$, $\boldsymbol{\mu} \in \mathbb{R}^n$, $\Sigma \in \mathbb{R}^{n\times n}$ *be a covariance matrix and* $\mathcal{U}$ *be an n-dimensional Thorin measure. The following are equivalent:*

(i) $\mathbf{Y} \sim VGG^n(\mathbf{d}, \boldsymbol{\mu}, \Sigma, \mathcal{U})$*;*

(ii) $\mathbf{Y} \sim L^n(\mathbf{m}_2, \mathbf{d} \diamond \Sigma, \mathcal{Y})$*, where*

$$\mathbf{m}_2 = \mathbf{d} \diamond \boldsymbol{\mu} + \int_{\mathbb{D}_*} \mathbf{x}\,\mathcal{Y}(\mathrm{d}\mathbf{x}), \tag{3.2.2}$$

$$\mathcal{Y}(\mathrm{d}\mathbf{x}) = \int_{[0,\infty)^n_*} \mathcal{V}_{\|\mathbf{u}\|^2,\mathbf{u}\diamond\boldsymbol{\mu},\mathbf{u}\diamond\Sigma}(\mathrm{d}\mathbf{x})\frac{\mathcal{U}(\mathrm{d}\mathbf{u})}{\|\mathbf{u}\|^2}; \tag{3.2.3}$$

(iii) $\mathbf{Y}$ *is an n-dimensional Lévy process with characteristic exponent*

$$\begin{aligned}\Psi_{\mathbf{Y}}(\boldsymbol{\theta}) &= \mathrm{i}\langle \mathbf{d}\diamond\boldsymbol{\mu},\boldsymbol{\theta}\rangle - \frac{1}{2}\|\boldsymbol{\theta}\|^2_{\mathbf{d}\diamond\Sigma} \\ &- \int_{[0,\infty)^n_*} \ln\left(1 - \frac{\mathrm{i}\langle \mathbf{u}\diamond\boldsymbol{\mu},\boldsymbol{\theta}\rangle}{\|\mathbf{u}\|^2} + \frac{\|\boldsymbol{\theta}\|^2_{\mathbf{u}\diamond\Sigma}}{2\|\mathbf{u}\|^2}\right)\mathcal{U}(\mathrm{d}\mathbf{u}), \quad \boldsymbol{\theta}\in\mathbb{R}^n.\end{aligned} \tag{3.2.4}$$

If (i)–(iii) are satisfied and Σ *is invertible, then* $\mathcal{Y} = \sum_J \mathcal{Y}_J$, $\mathcal{Y}_J(\mathbb{R}^n_*\backslash V_J) = 0$, *where* $\mathcal{Y}_J$ *is a Lévy measure which is absolutely continuous with respect to* $\mathcal{L}_J$, *having Lévy density*

$$\frac{\mathrm{d}\mathcal{Y}_J}{\mathrm{d}\mathcal{L}_J}(\mathbf{v}) = \int_{C_J} \nu_J(\mathbf{v},\mathbf{u})\mathcal{U}(\mathrm{d}\mathbf{u}), \tag{3.2.5}$$

where

$$\begin{aligned}\nu_J(\mathbf{v},\mathbf{u}) := c_J &\frac{\exp\big(\langle \mathbf{v},\mathbf{u}\diamond\boldsymbol{\mu}\rangle_{(\mathbf{u}\diamond\Sigma)^{-1}_J}\big)}{\|\mathbf{v}\|^{\#J}_{(\mathbf{u}\diamond\Sigma)^{-1}_J}|\mathbf{u}\diamond\Sigma|^{1/2}_J} \\ &\times \mathfrak{K}_{\#J/2}\Big(\big(2\|\mathbf{u}\|^2 + \|\mathbf{u}\diamond\boldsymbol{\mu}\|^2_{(\mathbf{u}\diamond\Sigma)^{-1}_J}\big)^{1/2}\|\mathbf{v}\|_{(\mathbf{u}\diamond\Sigma)^{-1}_J}\Big),\end{aligned} \tag{3.2.6}$$

$c_J := 2/(2\pi)^{\#J/2}$, $\mathbf{v}\in V_J$ *and* $\mathbf{u}\in C_J$.

Proof. The statements in Parts (i)–(iii) characterise the law of $\mathbf{Y}$, so it suffices to prove only one direction.

(i) ⇔ *(ii).* The formulas of the triplet $(\mathbf{m}_2, \mathbf{d}\diamond\Sigma, \mathcal{Y})$ follow from Proposition 2.3.4. In particular, by (2.3.5),

$$\mathbf{m}_2 = \mathbf{d}\diamond\boldsymbol{\mu} + \int_{[0,\infty)^n_*} \mathbb{E}[\mathbf{B}(\mathbf{t})\mathbf{1}_{\mathbb{D}}(\mathbf{B}(\mathbf{t}))]\,\mathcal{T}(\mathrm{d}\mathbf{t}).$$

Using (2.3.7) and the fact that $\mathbf{m}_2$ must be finite, we have

$$\int_{[0,\infty)^n_*} \mathbb{E}[\mathbf{B}(\mathbf{t})\mathbf{1}_{\mathbb{D}}(B(\mathbf{t}))]\,\mathcal{T}(\mathrm{d}\mathbf{t}) = \int_{\mathbb{D}_*} \mathbf{x}\,\mathcal{Y}(\mathrm{d}\mathbf{x}), \tag{3.2.7}$$

where both integrals are finite. This proves (3.2.2).

Let $A\subseteq\mathbb{R}^n_*$ be a Borel set. For $\mathbf{B}\sim BM^n(\boldsymbol{\mu},\Sigma)$ and $\mathbf{B}^{(\mathbf{u})}\sim BM^n(\mathbf{u}\diamond\boldsymbol{\mu}\,\mathbf{u}\diamond\Sigma)$, $\mathbf{u}\in[0,\infty)^n_*$, Proposition 1.2.2 implies $\mathbf{B}(g\mathbf{u}) \stackrel{D}{=} \mathbf{B}^{(\mathbf{u})}(g)$, $g\geq 0$, and combining this with (1.3.3) gives

$$\int_{(0,\infty)} \mathbb{P}(\mathbf{B}(g\mathbf{u}) \in A)\, \mathcal{G}_{\|\mathbf{u}\|^2}(\mathrm{d}g) = \mathcal{V}_{\|\mathbf{u}\|^2,\mathbf{u}\diamond\boldsymbol{\mu},\mathbf{u}\diamond\Sigma}(A).$$

Thus, by (2.3.7) and (3.1.6), we have

$$\mathcal{Y}(A) = \int_{[0,\infty)^n_*} \int_{(0,\infty)} \mathbb{P}(\mathbf{B}(g\mathbf{u}) \in A)\, \mathcal{G}_{\|\mathbf{u}\|^2}(\mathrm{d}g) \frac{\mathcal{U}(\mathrm{d}\mathbf{u})}{\|\mathbf{u}\|^2} \tag{3.2.8}$$

$$= \int_{[0,\infty)^n_*} \mathcal{V}_{\|\mathbf{u}\|^2,\mathbf{u}\diamond\boldsymbol{\mu},\mathbf{u}\diamond\Sigma}(A)\, \frac{\mathcal{U}(\mathrm{d}\mathbf{u})}{\|\mathbf{u}\|^2}, \tag{3.2.9}$$

which matches the RHS of (3.2.3) evaluated at A.

(i) ⇔ (iii). Since the integrals in (3.2.7) must be finite, using the Lévy-Khintchine formula (1.1.2) with the characteristics in Part (i) gives

$$\Psi_{\mathbf{Y}}(\boldsymbol{\theta}) = \mathrm{i}\langle \mathbf{d}\diamond\boldsymbol{\mu}, \boldsymbol{\theta}\rangle - \frac{1}{2}\|\boldsymbol{\theta}\|^2_{\mathbf{d}\diamond\Sigma} + I(\boldsymbol{\theta}), \tag{3.2.10}$$

where using (3.2.3) gives

$$I(\boldsymbol{\theta}) := \int_{\mathbb{R}^n_*} (e^{\mathrm{i}\langle\boldsymbol{\theta},\mathbf{x}\rangle} - 1)\, \mathcal{Y}(\mathrm{d}\mathbf{x})$$

$$= \int_{[0,\infty)^n_*} \int_{\mathbb{R}^n_*} (e^{\mathrm{i}\langle\boldsymbol{\theta},\mathbf{x}\rangle} - 1)\, \mathcal{V}_{\|\mathbf{u}\|^2,\mathbf{u}\diamond\boldsymbol{\mu},\mathbf{u}\diamond\Sigma}(\mathrm{d}\mathbf{x}) \frac{\mathcal{U}(\mathrm{d}\mathbf{u})}{\|\mathbf{u}\|^2}. \tag{3.2.11}$$

Now Proposition 1.3.4 (ii), followed by (1.3.4) implies that

$$\int_{\mathbb{R}^n_*} (e^{\mathrm{i}\langle\boldsymbol{\theta},\mathbf{x}\rangle} - 1)\, \mathcal{V}_{\|\mathbf{u}\|^2,\mathbf{u}\diamond\boldsymbol{\mu},\mathbf{u}\diamond\Sigma}(\mathrm{d}\mathbf{x}) = \Psi_{\mathbf{V}}(\boldsymbol{\theta})$$

$$= -\|\mathbf{u}\|^2 \ln\left(1 - \frac{\mathrm{i}\langle \mathbf{u}\diamond\boldsymbol{\mu}, \boldsymbol{\theta}\rangle}{\|\mathbf{u}\|^2} + \frac{\|\boldsymbol{\theta}\|^2_{\mathbf{u}\diamond\Sigma}}{2\|\mathbf{u}\|^2}\right).$$

Combining this with (3.2.10) and (3.2.11) completes the proof of (3.2.4).

Lévy density. Let $\emptyset \neq J = \{j_1, \dots, j_m\} \subseteq \{1, \dots, n\}$, $\mathbf{u} \in C_J$ and $A \subseteq \mathbb{R}^n_*$ be a Borel set. Recall that $(\mathbf{u}\diamond\Sigma)^{-1}_J$ exists and $|\mathbf{u}\diamond\Sigma|_J > 0$ because Σ is invertible.

As in Remark 3.2.5, let $\mathcal{Y}_J$ be the Lévy measure of $\mathbf{Y}_J \overset{D}{=} \mathbf{B} \odot \mathbf{T}_J$. Let $\boldsymbol{\pi}^J : \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{x} \mapsto \mathbf{x}\boldsymbol{\pi}^J := (\langle \mathbf{x}, \mathbf{e}_j\rangle)_{j\in J}$. Let $(\mathbf{u}\diamond\boldsymbol{\mu})^J := (\mathbf{u}\diamond\boldsymbol{\mu})\boldsymbol{\pi}^J \in \mathbb{R}^m$ and $(\mathbf{u}\diamond\Sigma)^J \in \mathbb{R}^{m\times m}$ be the invertible principal submatrix of $\mathbf{u}\diamond\Sigma$ formed by keeping only the rows and columns in the index set J. For $\mathbf{v} = (v_1, \dots, v_n) \in V_J$, let $\mathbf{v}^J := (v_{j_1}, \dots, v_{j_m})$ and $\mathrm{d}\mathbf{v}^J$ be the Lebesgue measure on $\mathbb{R}^m$. Using (1.3.3) followed by (1.3.5), when $\mathbf{u} \in C_J$, we have

$$\mathcal{V}_{\|\mathbf{u}\|^2,\mathbf{u}\diamond\boldsymbol{\mu},\mathbf{u}\diamond\Sigma}(A) = \mathcal{V}_{\|\mathbf{u}\|^2,(\mathbf{u}\diamond\boldsymbol{\mu})^J,(\mathbf{u}\diamond\Sigma)^J}((A\cap V_J)\boldsymbol{\pi}^J)$$

$$= \int_{\mathbb{R}^m_*} \mathbf{1}_{(A\cap V_J)\boldsymbol{\pi}^J}(\mathbf{v}^J) \frac{\mathrm{d}\mathcal{V}_{\|\mathbf{u}\|^2,(\mathbf{u}\diamond\boldsymbol{\mu})^J,(\mathbf{u}\diamond\Sigma)^J}}{\mathrm{d}\mathbf{v}^J}(\mathbf{v}^J)\,\mathrm{d}\mathbf{v}^J. \qquad (3.2.12)$$

Note that $\mathbf{1}_{(A\cap V_J)\boldsymbol{\pi}^J}(\mathbf{v}^J) = \mathbf{1}_{A\cap V_J}(\mathbf{v})$ for all $\mathbf{v} \in V_J$ and $(\mathrm{d}\mathcal{V}_{\|\mathbf{u}\|^2,(\mathbf{u}\diamond\boldsymbol{\mu})^J,(\mathbf{u}\diamond\Sigma)^J}/\mathrm{d}\mathbf{v}^J)(\mathbf{v}^J)$ $= \|\mathbf{u}\|^2\nu_J(\mathbf{v},\mathbf{u})$ for all $\mathbf{u} \in C_J$ and $\mathbf{v} \in V_J$. Therefore, we can write the integral in (3.2.12) over all components of $\mathbf{v}$ as

$$\mathcal{V}_{\|\mathbf{u}\|^2,\mathbf{u}\diamond\boldsymbol{\mu},\mathbf{u}\diamond\Sigma}(A) = \int_{V_J} \mathbf{1}_{A\cap V_J}(\mathbf{v})\|\mathbf{u}\|^2\nu_J(\mathbf{v},\mathbf{u})\,\mathcal{L}_J(\mathrm{d}\mathbf{v}), \quad \mathbf{u} \in C_J. \qquad (3.2.13)$$

Now using (3.2.9), followed by (3.2.13), we have

$$\begin{aligned}\mathcal{Y}_J(A) &= \int_{C_J} \mathcal{V}_{\|\mathbf{u}\|^2,\mathbf{u}\diamond\boldsymbol{\mu},\mathbf{u}\diamond\Sigma}(A)\,\frac{\mathcal{U}(\mathrm{d}\mathbf{u})}{\|\mathbf{u}\|^2}\\ &= \int_{V_J} \mathbf{1}_A(\mathbf{v})\left(\int_{C_J} \nu_J(\mathbf{v},\mathbf{u})\,\mathcal{U}(\mathrm{d}\mathbf{u})\right)\mathcal{L}_J(\mathrm{d}\mathbf{v}),\end{aligned}$$

which proves the Lévy density formula of $\mathcal{Y}_J$ with respect to $\mathcal{L}_J$. Also, we see that $\mathcal{Y}_J(\mathbb{R}^n_*\backslash V_J) = 0$. Meanwhile, by the independence of $\mathbf{Y}_J$ for all $\emptyset \neq J \subseteq \{1,\ldots,n\}$, we have $\mathcal{Y}(A) = \sum_J \mathcal{Y}_J(A)$ as required. □

In Theorems 2.3 and 2.5 of [BKMS17], formulas for the characteristic exponents and Lévy density of $VGG^{n,1}$ and $VGG^{n,n}$ processes are stated separately, while Theorem 3.2.6 here unifies both classes as special cases.

3.3 Sample Path Properties

To see how sample path properties such as the q-variation of the Thorin subordinator is propagated through Brownian motion, we generalise Propositions 2.6 and 2.7 in [BKMS17], which gave a corresponding result in the context of $VGG^{n,1}$ and $VGG^{n,n}$ processes. The q-variation is related to the jump activity of the Lévy process as noted in Section 2.3.6, where we discussed the sample path properties of weakly subordinated processes in general.

Recall that $\mathbb{D}^+ := \mathbb{D} \cap [0,\infty)^n$.

Proposition 3.3.1. *Let* $\mathbf{T} \sim GGC^n_S(\mathbf{d},\mathcal{U})$ *and* $\mathbf{Y} \sim VGG^n(\mathbf{d},\boldsymbol{\mu},\Sigma,\mathcal{U})$ *with Lévy measures* $\mathcal{T}$ *and* $\mathcal{Y}$*, respectively. Suppose* $0 < q < 1$.

(i) $\int_{\mathbb{D}^C} \mathcal{U}(\mathrm{d}\mathbf{u})/\|\mathbf{u}\|^q < \infty$ *if and only if* $\int_{\mathbb{D}^+_*} \|\mathbf{t}\|^q\,\mathcal{T}(\mathrm{d}\mathbf{t}) < \infty$.

(ii) If $\int_{\mathbb{D}^C} \mathcal{U}(\mathrm{d}\mathbf{u})/\|\mathbf{u}\|^q < \infty$*, then* $\int_{\mathbb{D}_*} \|\mathbf{x}\|^{2q}\,\mathcal{Y}(\mathrm{d}\mathbf{x}) < \infty$*. If* Σ *is invertible, then the converse also holds.*

(iii) If $\mathbf{d} = \mathbf{0}$ *and* $\int_{\mathbb{D}^C} \mathcal{U}(\mathrm{d}\mathbf{u})/\|\mathbf{u}\|^{1/2} < \infty$*, then* $\mathbf{Y} \sim FV^n$*. If* Σ *is invertible, then the converse also holds. Also, if* $\mathbf{d} = \mathbf{0}$ *and* $\mathbf{Y} \sim FV^n$*, then* $\mathbf{Y}$ *is driftless.*

Proof. (i). See Proposition 2.6 (a) in [BKMS17].

(ii). Let $0 < q < 1$. Let $\mathbf{B} \sim BM^n(\boldsymbol{\mu}, \Sigma)$, $\Sigma = (\Sigma_{kl}) \in \mathbb{R}^{n \times n}$ is a covariance matrix. For $\mathbf{t} \in [0, \infty)^n_*$, let $\psi(\mathbf{t}) := \mathbb{E}[\|\mathbf{B}(\mathbf{t})\|^{2q} \mathbf{1}_{\mathbb{D}_*}(\mathbf{B}(\mathbf{t}))]$.

Sufficiency. For $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{R}^n$, set $|\mathbf{z}| := (|z_1|, \ldots, |z_n|)$. Set $|\Sigma| := (|\Sigma_{kl}|) \in \mathbb{R}^{n \times n}$. Let $\mathbf{t} \in [0, \infty)^n_*$. By the eigendecomposition of symmetric matrices, we can write $\mathbf{t} \diamond \Sigma = UDU'$, where $U \in \mathbb{R}^{n \times n}$ is an orthogonal matrix and $D \in \mathbb{R}^{n \times n}$ is a diagonal matrix. Thus,

$$\begin{aligned} \|\mathbf{z}(\mathbf{t} \diamond \Sigma)^{1/2}\|^2 &= (\mathbf{z}UD^{1/2}U')(\mathbf{z}UD^{1/2}U')' \\ &= \|\mathbf{z}\|^2_{\mathbf{t} \diamond \Sigma} \\ &\leq \|\mathbf{t}\|_\infty \sum_{k=1}^n \sum_{l=1}^n |z_k||z_l||\Sigma_{kl}| \\ &\leq \|\mathbf{t}\| \||\mathbf{z}|\|^2_{|\Sigma|} \end{aligned} \tag{3.3.1}$$

by (1.2.5). Similarly,

$$\|\mathbf{t} \diamond \boldsymbol{\mu}\| \leq \|\boldsymbol{\mu}\| \|\mathbf{t}\|. \tag{3.3.2}$$

Let $\mathbf{Z} := (Z_1, \ldots, Z_n)$ be standard normal random vector on $\mathbb{R}^n$ and note that

$$\psi(\mathbf{t}) \leq \mathbb{E}[\|\mathbf{B}(\mathbf{t})\|^{2q}] = \mathbb{E}[\|\mathbf{t} \diamond \boldsymbol{\mu} + \mathbf{Z}(\mathbf{t} \diamond \Sigma)^{1/2}\|^{2q}] \leq \mathbb{E}[(\|\mathbf{t} \diamond \boldsymbol{\mu}\| + \|\mathbf{Z}(\mathbf{t} \diamond \Sigma)^{1/2}\|)^{2q}].$$

due to Example 1.2.3 and the triangle inequality. Applying the C_r inequality (see Equation (1) in [vBE65]) with the constant $2^{(2q-1)^+} < 2$ to the RHS of the above display gives

$$\psi(\mathbf{t}) \leq 2(\|\mathbf{t} \diamond \boldsymbol{\mu}\|^{2q} + \mathbb{E}[\|\mathbf{Z}(\mathbf{t} \diamond \Sigma)^{1/2}\|^{2q}])$$

Combining this with (3.3.1) and (3.3.2) yields

$$\begin{aligned} \psi(\mathbf{t}) &\leq 2(\|\boldsymbol{\mu}\|^{2q}\|\mathbf{t}\|^{2q} + \mathbb{E}[\||\mathbf{Z}|\|^{2q}_{|\Sigma|}]\|\mathbf{t}\|^q) \\ &\leq C_1(\|\mathbf{t}\|^{2q} \vee \|\mathbf{t}\|^q). \end{aligned}$$

with the finite constant $C_1 := 4(\|\boldsymbol{\mu}\|^{2q} + \mathbb{E}[\||\mathbf{Z}|\|^{2q}_{|\Sigma|}])$. Also, $\psi(\mathbf{t}) \leq 1$, so that

$$\psi(\mathbf{t}) \leq (1 + C_1)(1 \wedge (\|\mathbf{t}\|^{2q} \vee \|\mathbf{t}\|^q)) = (1 + C_1)(1 \wedge \|\mathbf{t}\|^q), \tag{3.3.3}$$

for all $\mathbf{t} \in [0,\infty)^n_*$.

Finally, by (2.3.7) and then (3.3.3), we have

$$\begin{aligned}\int_{\mathbb{D}_*} \|\mathbf{x}\|^{2q}\,\mathcal{Y}(\mathrm{d}\mathbf{x}) &= \int_{[0,\infty)^n_*} \psi(\mathbf{t})\,\mathcal{T}(\mathrm{d}\mathbf{t}) \\ &\le (1+C_1)\Big(\int_{\mathbb{D}_*} \|\mathbf{t}\|^q\,\mathcal{T}(\mathrm{d}\mathbf{t}) + \mathcal{T}(\mathbb{D}^C)\Big),\end{aligned} \tag{3.3.4}$$

where the first term on the RHS is finite due to Part (i) and second term is finite due to (1.1.3).

Necessity. The proof is completed provided we can show that

$$i := \inf_{\mathbf{t}\in\mathbb{D}^+_*} \frac{\psi(\mathbf{t})}{\|\mathbf{t}\|^q} > 0.$$

This would imply $\psi(\mathbf{t}) \ge i\|\mathbf{t}\|^q$ for all $\mathbf{t} \in \mathbb{D}^+_*$, giving

$$\int_{\mathbb{D}^+_*} \|\mathbf{t}\|^q\,\mathcal{T}(\mathrm{d}\mathbf{t}) \le \frac{1}{i}\int_{\mathbb{D}^+_*} \psi(\mathbf{t})\,\mathcal{T}(\mathrm{d}\mathbf{t}) \le \frac{1}{i}\int_{[0,\infty)^n_*} \psi(\mathbf{t})\,\mathcal{T}(\mathrm{d}\mathbf{t}) = \frac{1}{i}\int_{\mathbb{D}_*} \|\mathbf{x}\|^{2q}\,\mathcal{Y}(\mathrm{d}\mathbf{x}),$$

where the last equality follows from (3.3.4). Since the RHS is finite by assumption, Part (i) would then imply $\int_{\mathbb{D}^C} \mathcal{U}(\mathrm{d}\mathbf{u})/\|\mathbf{u}\|^q < \infty$ as required.

Let $\phi(\mathbf{t}) := \psi(\mathbf{t})/\|\mathbf{t}\|^q$, for $\mathbf{t} \in [0,\infty)^n_*$. There exists a sequence $\mathbf{t}_m \to \mathbf{t}_0$ with $\mathbf{t}_0 \in \mathbb{D}^+$, $\mathbf{t}_m \in \mathbb{D}^+_*$, $m \in \mathbb{N}$, such that $\phi(\mathbf{t}_m) \to i$ as $m \to \infty$. Moreover, $\mathbf{s}_m := \mathbf{t}_m/\|\mathbf{t}_m\| \to \mathbf{s}_0$ as $m \to \infty$ for some $\mathbf{s}_0 \in \mathbb{S}_+ := \mathbb{S} \cap [0,\infty)^n$.

If $\mathbf{t}_0 \ne \mathbf{0}$, then we find $\emptyset \ne J \subseteq \{1,\dots,n\}$ such that $\mathbf{t}_0 \in C_J$. Now we have

$$i = \lim_{m\to\infty} \frac{\psi(\mathbf{t}_m)}{\|\mathbf{t}_m\|^q} \tag{3.3.5}$$

$$\ge \Big(\liminf_{m\to\infty} \frac{\psi(\mathbf{t}_m)}{\|\mathbf{t}_0\|^q}\Big)\Big(\liminf_{m\to\infty} \frac{\|\mathbf{t}_0\|^q}{\|\mathbf{t}_m\|^q}\Big)$$

$$= \frac{1}{\|\mathbf{t}_0\|^q} \liminf_{m\to\infty} \psi(\mathbf{t}_m) \tag{3.3.6}$$

Let $\psi^*(\mathbf{t}) := \mathbb{E}[\|\mathbf{B}(\mathbf{t})\|^{2q}\mathbf{1}_{(0,1)}(\|\mathbf{B}(\mathbf{t})\|)]$, $\mathbf{t} \in [0,\infty)^n_*$ and note that $\psi^*(\mathbf{t}) = \psi(\mathbf{t})$ because $\mathbb{P}(\|\mathbf{B}(\mathbf{t})\| = 1) = 0$ as the distribution of $\mathbf{B}(\mathbf{t})$ is absolutely continuous. Thus, by Fatou's lemma (see Lemma A.3.2), we have

$$\begin{aligned}\liminf_{m\to\infty} \psi(\mathbf{t}_m) &= \liminf_{m\to\infty} \psi^*(\mathbf{t}_m) \\ &\ge \mathbb{E}\Big[\liminf_{m\to\infty} \|\mathbf{B}(\mathbf{t}_m)\|^{2q}\mathbf{1}_{(0,1)}(\|\mathbf{B}(\mathbf{t}_m)\|)\Big] \\ &\ge \psi^*(\mathbf{t}_0),\end{aligned} \tag{3.3.7}$$

where the last line follows since $\mathbf{t} \mapsto \|\mathbf{B}(\mathbf{t})\|^{2q}\mathbf{1}_{(0,1)}(\|\mathbf{B}(\mathbf{t})\|)$ is a lower semi-continuous function at $\mathbf{t} = \mathbf{t}_0$ as $\mathbf{B}$ has continuous sample paths. Recall that $\mathbf{B}(\mathbf{t}_0) \sim N(\mathbf{t}_0 \diamond \boldsymbol{\mu}, \mathbf{t}_0 \diamond \Sigma)$ by (1.2.11). Since Σ is invertible, combining Lemma A.2.2 (iii) and $\mathbf{t}_0 \neq \mathbf{0}$ implies $\mathbf{t}_0 \diamond \Sigma \neq 0$. Thus, $\mathbb{P}(0 < \|\mathbf{B}(\mathbf{t}_0)\| < 1) > 0$, which implies

$$\psi^*(\mathbf{t}_0) = \int_{\{0<\|\mathbf{x}\|<1\}} \|\mathbf{x}\|^{2q}\, \mathbb{P}(\mathbf{B}(\mathbf{t}_0) \in \mathrm{d}\mathbf{x}) > 0$$

since the integrand is strictly positive (see Theorem 13.2 in [Bau92]). Finally, combining this with (3.3.6) and (3.3.7) gives $i > 0$.

Now if $\mathbf{t}_0 = \mathbf{0}$, let $\mathbf{B}^* \sim BM^n(\mathbf{0}, \Sigma)$ and $\mathbf{W}_m := (\|\mathbf{t}_m\|^{1/2}\mathbf{s}_m) \diamond \boldsymbol{\mu} + \mathbf{B}^*(\mathbf{s}_m)$, $m \in \mathbb{N}$. Recalling that $\mathbf{Z} := (Z_1, \ldots, Z_n)$ is a standard normal random vector on $\mathbb{R}^n$, we have

$$\mathbf{B}(\mathbf{t}_m) \stackrel{D}{=} \mathbf{t}_m \diamond \boldsymbol{\mu} + \mathbf{Z}(\mathbf{t}_m \diamond \Sigma)^{1/2} \stackrel{D}{=} \|\mathbf{t}_m\|^{1/2}\mathbf{W}_m.$$

By noting (3.3.5) and $\mathbb{P}(\mathbf{W}_m = \mathbf{0}) = 0$, this implies

$$i \geq \liminf_{m\to\infty} \mathbb{E}[\|\mathbf{W}_m\|^{2q}\mathbf{1}_{\mathbb{D}}(\|\mathbf{t}_m\|^{1/2}\mathbf{W}_m)] \geq \mathbb{E}\Big[\liminf_{m\to\infty} \|\mathbf{W}_m\|^{2q}\mathbf{1}_{\mathbb{D}}(\|\mathbf{t}_m\|^{1/2}\mathbf{W}_m)\Big]$$

by Fatou's lemma (see Lemma A.3.2). In addition, as $m \to \infty$, we have $\mathbf{W}_m \xrightarrow{\text{a.s.}} \mathbf{B}^*(\mathbf{s}_0)$ and $\mathbf{1}_{\mathbb{D}}(\|\mathbf{t}_m\|^{1/2}\mathbf{W}_m) \xrightarrow{\text{a.s.}} 1$ since $\mathbf{t}_0 = \mathbf{0}$. Thus, $i \geq \mathbb{E}[\|\mathbf{B}^*(\mathbf{s}_0)\|^{2q}] > 0$, which completes the proof.

(iii). Sufficiency. Using Part (ii) with $q = 1/2$, $\int_{\mathbb{D}^C} \mathcal{U}(\mathrm{d}\mathbf{u})/\|\mathbf{u}\|^{1/2} < \infty$ implies (1.1.4). If $\mathbf{d} = \mathbf{0}$, then $\mathbf{d} \diamond \Sigma = 0$ in Theorem 3.2.6 (ii). Thus, $\mathbf{Y} \sim FV^n$ by Proposition 1.1.11.

Necessity. Since $\mathbf{Y} \sim FV^n$, Proposition 1.1.11 implies that $\mathbf{d} \diamond \Sigma = 0$. We have $\mathrm{diag}(\mathbf{d} \diamond \Sigma) = \mathbf{d} \diamond \mathrm{diag}(\Sigma) = \mathbf{0}$, which implies $\mathbf{d} = \mathbf{0}$ because $\Sigma_{kk} > 0$ for all $1 \leq k \leq n$ by Lemma A.2.2 (iii) and the invertibility of Σ . For $\mathbf{Y} \sim FV^n$, Proposition 1.1.11 also implies that (1.1.4) holds, which implies by Part (ii), with $q = 1/2$, and the invertibility of Σ, that $\int_{\mathbb{D}^C} \mathcal{U}(\mathrm{d}\mathbf{u})/\|\mathbf{u}\|^{1/2}$.

Driftlessness. If $\mathbf{d} = \mathbf{0}$, then the drift of $\mathbf{Y}$ is $\mathbf{m}_2 - \int_{\mathbb{D}_*} \mathbf{x}\, \mathcal{Y}(\mathrm{d}\mathbf{x}) = \mathbf{0}$, where $\mathbf{m}_2$ is given in (3.2.2) and the finiteness of the integral is ensured by $\mathbf{Y} \sim FV^n$. $\square$

Chapter 4

Weak Variance-Alpha-Gamma Processes

Our second major application of weak subordination is to construct weak variance-alpha-gamma processes, a generalisation of the variance-alpha-gamma processes of Semeraro [Sem08]. The latter is a constructed using multivariate subordination and a Brownian motion subordinate with independent components. We use weak subordination to create a weakly subordinated counterpart to this process, which we call a weak variance-alpha-gamma process. This allows for more flexible dependence modelling as the Brownian motion may now have dependent components. Several results in previous chapters are applied to this class of processes.

In Section 4.1, we review the definition of variance-alpha-gamma processes and introduce weak variance-alpha-gamma processes. In Section 4.2, we show that weak variance-alpha-gamma processes are VGG^n processes and we characterise their laws. In Section 4.3, we derive some useful properties of weak variance-alpha-gamma processes, including moment formulas. In Section 4.4, we show that these processes can be decomposed into a sum of independent variance-gamma processes. In Section 4.5, a condition for Fourier invertibility is given. This has important applications in calibrating parameters with maximum likelihood as the density function is not explicitly known. In Section 4.6, we investigate methods for calibrating weak variance-alpha-gamma processes to discretely observed data. We apply method of moments, digital moment estimation and maximum likelihood to both simulated and financial data, and discuss our findings.

4.1 Construction

Recall that the variance-gamma process $\mathbf{V} \stackrel{D}{=} \mathbf{B} \circ (G\mathbf{e})$ is constructed using univariate subordination, where $\mathbf{B} \sim BM^n(\boldsymbol{\mu}, \Sigma)$, $G \sim \Gamma_S(b)$. As such, it has a common time change, but it cannot have idiosyncratic time changes. The jumps of the subordinator $G\mathbf{e}$ causes jumps in all components of $\mathbf{V}$ provided that all the components of $\mathbf{B}$ are nonzero. This results in a restrictive dependence structure and a process with equal kurtosis in each component when the skewness parameter $\boldsymbol{\mu} = \mathbf{0}$.

To allow for more flexible multivariate dependence modelling, Semeraro [Sem08] introduced the alpha-gamma subordinator by taking the superposition of ray subordinators in a way that models both common and idiosyncratic time changes while maintaining variance-gamma marginal components when subordinated with Brownian motion. Here, we review some relevant definitions and properties, and then construct weak variance-alpha-gamma processes as an application of weak subordination to allow better dependence modelling.

Definition 4.1.1. Let $n \geq 2$. Let $a > 0$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in (0, 1/a)^n$ and

$$\beta_k := \frac{1 - a\alpha_k}{\alpha_k}, \quad 1 \leq k \leq n. \tag{4.1.1}$$

Let $G_0 \sim \Gamma_S(a, 1)$, $G_k \sim \Gamma_S(\beta_k, 1/\alpha_k)$, $1 \leq k \leq n$, be independent. An n-dimensional subordinator $\mathbf{T} \sim AG^n(a, \boldsymbol{\alpha})$ is an *alpha-gamma subordinator* if

$$\mathbf{T} = (T_1, \dots, T_n) \stackrel{D}{=} G_0\boldsymbol{\alpha} + (G_1, \dots, G_n). \tag{4.1.2}$$

Note that $\mathbf{T}$ is a Lévy process as it the linear transformation of independent Lévy processes $G_0, \dots, G_n$, and it is a subordinator as it is nondecreasing. The parameters in the definition were specified by Semeraro [Sem08] in such a way that the marginal components of $\mathbf{T}$ are standard gamma subordinators

$$T_k \sim \Gamma_S(1/\alpha_k), \quad 1 \leq k \leq n. \tag{4.1.3}$$

We give the Laplace exponent of an AG subordinator.

Lemma 4.1.2. *A subordinator* $\mathbf{T} \sim AG^n(a, \boldsymbol{\alpha})$ *has Laplace exponent*

$$\Lambda_{\mathbf{T}}(\boldsymbol{\lambda}) = a \ln(1 + \langle \boldsymbol{\lambda}, \boldsymbol{\alpha} \rangle) + \sum_{k=1}^{n} \beta_k \ln(1 + \alpha_k \langle \boldsymbol{\lambda}, \mathbf{e}_k \rangle), \quad \boldsymbol{\lambda} \in [0, \infty)^n. \tag{4.1.4}$$

Proof. Using (4.1.2) and the independence of $G_0, \dots, G_n$, the Laplace transform of $\mathbf{T}$ is

$$\phi_{\mathbf{T}}(\boldsymbol{\lambda}) = \mathbb{E}[\exp(-\langle \boldsymbol{\lambda}, \boldsymbol{\alpha} \rangle G_0(1))] \prod_{k=1}^{n} \mathbb{E}[\exp(-\langle \boldsymbol{\lambda}, \mathbf{e}_k \rangle G_k(1))]$$
$$= (1 + \langle \boldsymbol{\lambda}, \boldsymbol{\alpha} \rangle)^{-a} \prod_{k=1}^{n} (1 + \alpha_k \langle \boldsymbol{\lambda}, \mathbf{e}_k \rangle)^{-\beta_k}, \quad \boldsymbol{\lambda} \in [0, \infty)^n,$$

where the last line follows from (1.1.9). This implies the Laplace exponent given in (4.1.4). □

Remark 4.1.3. An AG subordinator $\mathbf{T}$ is often defined as in (4.1.2) but with $a, b > 0$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in (0, b/a)^n$, $G_0 \sim \Gamma_S(a, b)$, $G_k \sim \Gamma_S((b - a\alpha_k)/\alpha_k, b/\alpha_k)$, $1 \le k \le n$ (see [LS10, Sem08]). Using the same methods as the proof of Lemma 4.1.2, such a subordinator, denoted $\mathbf{T} \sim AG^n(a, b, \boldsymbol{\alpha})$, has Laplace exponent

$$\Lambda_{\mathbf{T}}(\boldsymbol{\lambda}) = a \ln\left(1 + \frac{\langle \boldsymbol{\lambda}, \boldsymbol{\alpha} \rangle}{b}\right) + \sum_{k=1}^{n} \frac{b - a\alpha_k}{\alpha_k} \ln\left(1 + \frac{\alpha_k \langle \boldsymbol{\lambda}, \mathbf{e}_k \rangle}{b}\right), \quad \boldsymbol{\lambda} \in [0, \infty)^n.$$

However, the laws of $AG^n(a, b, \boldsymbol{\alpha})$ and $AG^n(a, 1, \boldsymbol{\alpha}/b)$, for all $a, b > 0$, $\boldsymbol{\alpha} \in (0, b/a)^n$, are identical as they have the same Laplace exponent. Hence, we can always assume without loss of generality that $b = 1$, which explains our choice of parametrisation in Definition 4.1.1.

Another immediate consequence of Lemma 4.1.2 is that AG subordinators are Thorin subordinators and we can also determine their Lévy measure. Recall that $\mathcal{G}_{a,b}$ is defined in Definition 1.1.16.

Proposition 4.1.4. *Let $a > 0$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in (0, 1/a)^n$. The following are equivalent:*

(i) $\mathbf{T} \sim AG^n(a, \boldsymbol{\alpha})$;

(ii) $\mathbf{T} \sim GGC_S^n(\mathbf{0}, \mathcal{U}_{a,\boldsymbol{\alpha}})$, *where*

$$\mathcal{U}_{a,\boldsymbol{\alpha}} := a\boldsymbol{\delta}_{\boldsymbol{\alpha}/\|\boldsymbol{\alpha}\|^2} + \sum_{k=1}^{n} \beta_k \boldsymbol{\delta}_{\mathbf{e}_k/\alpha_k}; \tag{4.1.5}$$

(iii) $\mathbf{T} \sim S^n(\mathbf{0}, \mathcal{T}_{a,\boldsymbol{\alpha}})$, *where*

$$\mathcal{T}_{a,\boldsymbol{\alpha}} = \int_{(0,\infty)} \boldsymbol{\delta}_{g\boldsymbol{\alpha}}\, \mathcal{G}_{a,1}(\mathrm{d}g) + \sum_{k=1}^{n} \boldsymbol{\delta}_0^{\otimes(k-1)} \otimes \mathcal{G}_{\beta_k, 1/\alpha_k} \otimes \boldsymbol{\delta}_0^{\otimes(n-k)}. \tag{4.1.6}$$

Proof. The statements in Parts (i)–(iii) characterise the law of $\mathbf{T}$, so it suffices to prove only one direction.

(i) $\Leftrightarrow$ *(ii).* Note that $\mathcal{U}_{a,\boldsymbol{\alpha}}$ is finitely supported with $\|\boldsymbol{\alpha}/\|\boldsymbol{\alpha}\|^2\|, \|\mathbf{e}_k/\alpha_k\| > 0$, $1 \le k \le n$, so it satisfies (3.1.1) and the definition of a Thorin subordinator. Substituting $\mathbf{d} = \mathbf{0}$ and the Thorin measure $\mathcal{U}_{a,\boldsymbol{\alpha}}$ into the RHS of (3.1.2), we obtain the Laplace exponent of $\mathbf{T}$ in (4.1.4).

(i) $\Leftrightarrow$ *(iii).* Set $a_0 := a$, $b_0 := 1$, $\boldsymbol{\alpha}_0 := \boldsymbol{\alpha}$, $a_k := \beta_k$, $b_k := 1/\alpha_k$, $\boldsymbol{\alpha}_k := \mathbf{e}_k$, $1 \le k \le n$, then $\mathcal{U}_{a,\boldsymbol{\alpha}} = \sum_{k=0}^n a_k \boldsymbol{\delta}_{b_k\boldsymbol{\alpha}_k/\|\boldsymbol{\alpha}_k\|^2}$. Using (3.1.6) and (1.1.8), the associated Lévy measure evaluated at a Borel set $A \subseteq \mathbb{R}^n_*$ is

$$\begin{aligned}\mathcal{T}_{a,\boldsymbol{\alpha}}(A) &= \sum_{k=0}^n a_k \int_{(0,\infty)} \mathbf{1}_A\left(g\frac{b_k\boldsymbol{\alpha}_k}{\|\boldsymbol{\alpha}_k\|^2}\right) e^{-b_k^2 g/\|\boldsymbol{\alpha}_k\|^2}\,\frac{\mathrm{d}g}{g}\\ &= \sum_{k=0}^n \int_{(0,\infty)} \mathbf{1}_A(g\boldsymbol{\alpha}_k) a_k e^{-b_k^2 g}\,\frac{\mathrm{d}g}{g}\\ &= \sum_{k=0}^n \int_{(0,\infty)} \boldsymbol{\delta}_{g\boldsymbol{\alpha}_k}(A)\,\mathcal{G}_{a_k,b_k}(\mathrm{d}g),\end{aligned}$$

which matches (4.1.6) evaluated at A. Thus, $\mathbf{T} \sim S^n(\mathbf{0}, \mathcal{T}_{a,\boldsymbol{\alpha}})$. □

The Lévy measure $\mathcal{T}_{a,\boldsymbol{\alpha}}$ of an AG subordinator can alternatively be derived using Proposition 1.1.8, or Lemma 2.13 in [BKMS17], which gives the Lévy measure for a more general class of subordinators, or Equation (2.3) in [LS10] through its characteristic exponent. Now the variance-alpha-gamma process introduced in [LS10, Sem08] can be defined.

Definition 4.1.5. Let $n \ge 2$. Let $a > 0$, $\boldsymbol{\alpha} \in (0, 1/a)^n$, $\boldsymbol{\mu} \in \mathbb{R}^n$ and $\Sigma \in \mathbb{R}^{n\times n}$ be a diagonal covariance matrix. An n-dimensional Lévy process $\mathbf{Y} \sim VAG^n(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma)$ is a *variance-alpha-gamma process* if $\mathbf{Y} \overset{D}{=} \mathbf{B} \circ \mathbf{T}$, where $\mathbf{B} \sim BM^n(\boldsymbol{\mu}, \Sigma)$ and $\mathbf{T} \sim AG^n(a, \boldsymbol{\alpha})$ are independent.

Clearly, if $\mathbf{Y} \sim VAG^n(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma)$, then $\mathbf{Y} \sim VGG^{n,n}(\mathbf{0}, \boldsymbol{\mu}, \Sigma, \mathcal{U}_{a,\boldsymbol{\alpha}})$. While the VAG process incorporates both common and idiosyncratic time changes as a result of using an AG subordinator, it still has a restrictive dependence structure since the Brownian motion subordinate has independent components. To address this, we generalise the VAG process using weak subordination, which allows for the Brownian motion to have dependent components.

Definition 4.1.6. Let $n \ge 2$. Let $a > 0$, $\boldsymbol{\alpha} \in (0, 1/a)^n$, $\boldsymbol{\mu} \in \mathbb{R}^n$ and $\Sigma \in \mathbb{R}^{n\times n}$ be a covariance matrix. An n-dimensional Lévy process $\mathbf{Y} \sim WVAG^n(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma)$ is a *weak variance-alpha-gamma process* if $\mathbf{Y} \overset{D}{=} \mathbf{B} \odot \mathbf{T}$, where $\mathbf{B} \sim BM^n(\boldsymbol{\mu}, \Sigma)$ and $\mathbf{T} \sim AG^n(a, \boldsymbol{\alpha})$.

It is obvious that VAG processes are a subclass of $WVAG$ processes.

Lemma 4.1.7. *If* $\mathbf{Y} \sim WVAG^n(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma)$ *and* Σ *is diagonal, then* $\mathbf{Y} \sim VAG^n(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma)$.

Proof. Since $\mathbf{Y} \sim WVAG^n(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma)$, there exist independent $\mathbf{B} \sim BM^n(\boldsymbol{\mu}, \Sigma)$ and $\mathbf{T} \sim AG^n(a, \boldsymbol{\alpha})$ such that $\mathbf{Y} \stackrel{D}{=} \mathbf{B} \odot \mathbf{T} \stackrel{D}{=} \mathbf{B} \circ \mathbf{T}$, where the last equality follows from Theorem 2.3.5 and Σ being diagonal. Thus, $\mathbf{Y} \sim VAG^n(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma)$. □

4.2 Characteristics

Throughout this section and the next, we use the following notation. Let $a > 0$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in (0, 1/a)^n$, $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$, $\Sigma = (\Sigma_{kl}) \in \mathbb{R}^{n\times n}$ be a covariance matrix. Let $\mathbf{B} = (B_1, \dots, B_n) \sim BM^n(\boldsymbol{\mu}, \Sigma)$, $\mathbf{T} = (T_1, \dots, T_n) \sim AG^n(a, \boldsymbol{\alpha})$, $\mathbf{Y} = (Y_1, \dots, Y_n) \sim WVAG^n(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma)$.

Remark 4.2.1. Occasionally, we will use a setup involving the joint process $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2) \stackrel{D}{=} (\mathbf{T}, \mathbf{B} \odot \mathbf{T})$, which we set to be the semi-strong subordination of $\mathbf{B} \sim BM^n(\boldsymbol{\mu}, \Sigma)$ and $\mathbf{T} \sim AG^n(a, \boldsymbol{\alpha})$. So $\mathbf{T} = \mathbf{Z}_1$, and we set $\mathbf{Y} = \mathbf{Z}_2$. In particular, this implies that $\mathbf{Y} \stackrel{D}{=} \mathbf{B} \odot \mathbf{T}$ is the weak subordination of $\mathbf{B}$ and $\mathbf{T}$. We may assume without loss of generality that $\mathbf{Y} \sim WVAG^n(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma)$ is constructed and related to $\mathbf{T}$ in this way.

We now determine the characteristics of $WVAG$ processes by applying the results of Theorem 3.2.6. This also accounts for VAG processes as a special case when Σ is diagonal. The next proposition begins by noting that a $WVAG^n$ process is a VGG^n process, despite not necessarily being a $VGG^{n,1}$ or $VGG^{n,n}$ process.

Recall that $\mathcal{G}_{a,b}$, $\mathfrak{K}_\rho$, β_k, $\mathcal{U}_{a,\boldsymbol{\alpha}}$ and V_J are defined in (1.1.16), (1.3.2), (4.1.1), (4.1.5) and Remark 3.2.5, respectively.

Proposition 4.2.2. *Let* $n \geq 2$. *Let* $a > 0$, $\boldsymbol{\alpha} \in (0, 1/a)^n$, $\boldsymbol{\mu} \in \mathbb{R}^n$ *and* $\Sigma \in \mathbb{R}^{n\times n}$ *be a covariance matrix. Let* $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n_*$. *The following are equivalent:*

(i) $\mathbf{Y} \sim WVAG^n(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma)$;

(ii) $\mathbf{Y} \sim VGG^n(\mathbf{0}, \boldsymbol{\mu}, \Sigma, \mathcal{U}_{a,\boldsymbol{\alpha}})$;

(iii) $\mathbf{Y} \sim L^n(\mathbf{m}_2, 0, \mathcal{Y})$, *where*

$$\mathbf{m}_2 = \int_{\mathbb{D}_*} \mathbf{x}\, \mathcal{Y}(\mathrm{d}\mathbf{x}), \tag{4.2.1}$$

$$\begin{aligned}\mathcal{Y}(\mathrm{d}\mathbf{x}) = &\int_{(0,\infty)} \mathbb{P}(\mathbf{B}^{(\boldsymbol{\alpha})}(g) \in \mathrm{d}\mathbf{x})\, \mathcal{G}_{a,1}(\mathrm{d}g) + \sum_{k=1}^{n} \boldsymbol{\delta}_0^{\otimes(k-1)} \\ &\otimes \left(\int_{(0,\infty)} \mathbb{P}(B_k(g) \in \mathrm{d}x_k)\, \mathcal{G}_{\beta_k, 1/\alpha_k}(\mathrm{d}g) \right) \otimes \boldsymbol{\delta}_0^{\otimes(n-k)},\end{aligned} \tag{4.2.2}$$

and $\mathbf{B}^{(\boldsymbol{\alpha})} \sim BM^n(\boldsymbol{\alpha} \diamond \boldsymbol{\mu}, \boldsymbol{\alpha} \diamond \Sigma)$*;*

(iv) $\mathbf{Y}$ *is a Lévy process with characteristic exponent*

$$\begin{aligned}\Psi_{\mathbf{Y}}(\boldsymbol{\theta}) = & -a \ln\left(1 - \mathrm{i}\langle \boldsymbol{\alpha} \diamond \boldsymbol{\mu}, \boldsymbol{\theta}\rangle + \frac{1}{2}\|\boldsymbol{\theta}\|^2_{\boldsymbol{\alpha}\diamond\Sigma}\right) \\ & - \sum_{k=1}^{n} \beta_k \ln\left(1 - \mathrm{i}\alpha_k\mu_k\theta_k + \frac{1}{2}\alpha_k\Sigma_{kk}\theta_k^2\right),\end{aligned} \tag{4.2.3}$$

$\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$.

If (i)–(iv) are satisfied and Σ *is invertible, then* $\mathbf{Y}$ *has the Lévy measure*

$$\mathcal{Y}(\mathrm{d}\mathbf{x}) = \mathbf{1}_{(\mathbb{R}_*)^n}(\mathbf{x}) f_0(\mathbf{x})\mathrm{d}\mathbf{x} + \sum_{k=1}^{n} \mathbf{1}_{V_{\{k\}}}(\mathbf{x})(\boldsymbol{\delta}_0^{\otimes(k-1)} \otimes (f_k(x_k)\mathrm{d}x_k) \otimes \boldsymbol{\delta}_0^{\otimes(n-k)}) \tag{4.2.4}$$

on $\mathbb{R}^n_*$*, where the Lévy densities are*

$$\begin{aligned} f_0(\mathbf{v}) := & \frac{2a \exp(\langle \mathbf{v}, \boldsymbol{\alpha} \diamond \boldsymbol{\mu}\rangle_{(\boldsymbol{\alpha}\diamond\Sigma)^{-1}})}{(2\pi)^{n/2}|\boldsymbol{\alpha} \diamond \Sigma|^{1/2}\|\mathbf{v}\|^n_{(\boldsymbol{\alpha}\diamond\Sigma)^{-1}}} \\ & \times \mathfrak{K}_{n/2}((2 + \|\boldsymbol{\alpha} \diamond \boldsymbol{\mu}\|^2_{(\boldsymbol{\alpha}\diamond\Sigma)^{-1}})^{1/2}\|\mathbf{v}\|_{(\boldsymbol{\alpha}\diamond\Sigma)^{-1}}),\end{aligned} \tag{4.2.5}$$

$$f_k(v) := \frac{\beta_k}{|v|} \exp\left(\frac{v\alpha_k^{1/2}\mu_k - |v|(2\Sigma_{kk} + \alpha_k\mu_k^2)^{1/2}}{\alpha_k^{1/2}\Sigma_{kk}}\right) \tag{4.2.6}$$

for $\mathbf{v} \in (\mathbb{R}_*)^n$*,* $v \in \mathbb{R}_*$*.*

Proof. The statements in Parts (i)–(iv) characterise the law of $\mathbf{Y}$, so it suffices to prove only one direction.

(i) $\Leftrightarrow$ *(ii).* Since $\mathbf{Y} \sim WVAG^n(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma)$, we have $\mathbf{Y} \stackrel{D}{=} \mathbf{B} \odot \mathbf{T}$, where $\mathbf{B} \sim BM^n(\boldsymbol{\mu}, \Sigma)$ and $\mathbf{T} \sim GGC^n_S(\mathbf{0}, \mathcal{U}_{a,\boldsymbol{\alpha}})$ due to Lemma 4.1.4. It follows that $\mathbf{Y} \sim VGG^n(\mathbf{0}, \boldsymbol{\mu}, \Sigma, \mathcal{U}_{a,\boldsymbol{\alpha}})$ by definition.

(i) $\Leftrightarrow$ *(iii).* Assume the setup in Remark 4.2.1, and let $\mathcal{Z}$ be the Lévy measure of $\mathbf{Z}$. For all Borel sets $A \subseteq ([0, \infty)^n \times \mathbb{R}^n)_*$, using (2.2.5) and then (4.1.6), we have

$$\begin{aligned}\mathcal{Z}(A) &= \int_{[0,\infty)^n_*} \int_{\mathbb{R}^n} \mathbf{1}_A(\mathbf{t}, \mathbf{x})\, \mathbb{P}(\mathbf{B}(\mathbf{t}) \in \mathrm{d}\mathbf{x})\mathcal{T}_{a,\boldsymbol{\alpha}}(\mathrm{d}\mathbf{t}) \\ &= \int_{(0,\infty)} \int_{\mathbb{R}^n} \mathbf{1}_A(g\boldsymbol{\alpha}, \mathbf{x})\, \mathbb{P}(\mathbf{B}(g\boldsymbol{\alpha}) \in \mathrm{d}\mathbf{x})\mathcal{G}_{a,1}(\mathrm{d}g) \\ &\quad + \sum_{k=1}^{n} \int_{(0,\infty)} \int_{\mathbb{R}^n} \mathbf{1}_A(g\mathbf{e}_k, \mathbf{x})\, \mathbb{P}(\mathbf{B}(g\mathbf{e}_k) \in \mathrm{d}\mathbf{x})\mathcal{G}_{\beta_k,1/\alpha_k}(\mathrm{d}g)\end{aligned}$$

$$
\begin{aligned}
&= \int_{(0,\infty)} \int_{\mathbb{R}^n} \mathbf{1}_A(g\boldsymbol{\alpha}, \mathbf{x})\, \mathbb{P}(\mathbf{B}^{(\boldsymbol{\alpha})}(g) \in \mathrm{d}\mathbf{x}) \mathcal{G}_{a,1}(\mathrm{d}g) \\
&\quad + \sum_{k=1}^{n} \int_{(0,\infty)} \int_{\mathbb{R}} \mathbf{1}_A(g\mathbf{e}_k, x_k\mathbf{e}_k)\, \mathbb{P}(B_k(g) \in \mathrm{d}x_k) \mathcal{G}_{\beta_k,1/\alpha_k}(\mathrm{d}g),
\end{aligned} \tag{4.2.7}
$$

where the last line follows because $\mathbb{P}(\mathbf{B}(g\boldsymbol{\alpha}) \in \mathrm{d}\mathbf{x}) = \mathbb{P}(\mathbf{B}^{(\boldsymbol{\alpha})}(g) \in \mathrm{d}\mathbf{x})$ and $\mathbb{P}(\mathbf{B}(g\mathbf{e}_k) \in \mathrm{d}\mathbf{x}) = \boldsymbol{\delta}_0^{\otimes(k-1)} \otimes \mathbb{P}(B_k(g) \in \mathrm{d}x_k) \otimes \boldsymbol{\delta}_0^{\otimes(n-k)}$ for $g > 0$ and $1 \le k \le n$.

Applying the projection $\boldsymbol{\pi}_{\{n+1,\dots,2n\}}$ to the Lévy measure $\mathcal{Z}$ and using Proposition 1.1.8, the Lévy measure of $\mathbf{Y}$ satisfies

$$
\begin{aligned}
\mathcal{Y}(\widetilde{A}) &= \int_{(0,\infty)} \int_{\mathbb{R}^n} \mathbf{1}_{\widetilde{A}}(\mathbf{x})\, \mathbb{P}(\mathbf{B}^{(\boldsymbol{\alpha})}(g) \in \mathrm{d}\mathbf{x}) \mathcal{G}_{a,1}(\mathrm{d}g) \\
&\quad + \sum_{k=1}^{n} \int_{(0,\infty)} \int_{\mathbb{R}} \mathbf{1}_{\widetilde{A}}(x_k\mathbf{e}_k)\, \mathbb{P}(B_k(g) \in \mathrm{d}x_k) \mathcal{G}_{\beta_k,1/\alpha_k}(\mathrm{d}g)
\end{aligned}
$$

for all Borel set $\widetilde{A} \subseteq \mathbb{R}^n_*$, which matches (4.2.2) evaluated at $\widetilde{A}$. Then Theorem 3.2.6 (ii) shows that $\mathbf{Y} \sim L^n(\mathbf{m}_2, 0, \mathcal{Y})$ with $\mathbf{m}_2$ given by (4.2.1).

(i) ⇔ (iv). Since $\mathbf{Y} \sim VGG^n(\mathbf{0}, \boldsymbol{\mu}, \Sigma, \mathcal{U}_{a,\boldsymbol{\alpha}})$, the characteristic exponent is obtained by substituting $\mathbf{d} = \mathbf{0}$ and $\mathcal{U} = \mathcal{U}_{a,\boldsymbol{\alpha}}$ into (3.2.4).

Lévy density. Recalling the definition of C_J in Remark 3.2.5, the Thorin measure $\mathcal{U}_{a,\boldsymbol{\alpha}}$ given by (4.1.5) vanishes except on $C_{\{1,\dots,n\}}$ and $C_{\{k\}}$, $1 \le k \le n$. Assuming that Σ is invertible, for $\mathbf{x} \in \mathbb{R}^n_*$, the Lévy density formula in Theorem 3.2.6 allows the Lévy measure $\mathcal{Y}$ to be written as

$$
\begin{aligned}
\mathcal{Y}(\mathrm{d}\mathbf{x}) &= \mathbf{1}_{(\mathbb{R}_*)^n}(\mathbf{x}) a \nu_{\{1,\dots,n\}}\left(\mathbf{x}, \frac{\boldsymbol{\alpha}}{\|\boldsymbol{\alpha}\|^2}\right) \mathrm{d}\mathbf{x} \\
&\quad + \sum_{k=1}^{n} \mathbf{1}_{V_{\{k\}}}(\mathbf{x}) \beta_k \nu_{\{k\}}\left(\mathbf{x}, \frac{\mathbf{e}_k}{\alpha_k}\right) (\boldsymbol{\delta}_0^{\otimes(k-1)} \otimes \mathrm{d}x_k \otimes \boldsymbol{\delta}_0^{\otimes(n-k)}).
\end{aligned}
$$

Recalling (3.2.6), we have

$$
f_0(\mathbf{v}) := a\nu_{\{1,\dots,n\}}\left(\mathbf{v}, \frac{\boldsymbol{\alpha}}{\|\boldsymbol{\alpha}\|^2}\right) = \frac{2a}{(2\pi)^{n/2}} \frac{\exp(\mathfrak{E}_0(\mathbf{v}))}{\mathfrak{D}_0(\mathbf{v})} \mathfrak{K}_{n/2}(\mathfrak{A}_0(\mathbf{v})), \quad \mathbf{v} \in (\mathbb{R}_*)^n,
$$

where

$$
\begin{aligned}
\mathfrak{A}_0(\mathbf{v}) &= \left(\frac{2}{\|\boldsymbol{\alpha}\|^2} + \left\| \frac{\boldsymbol{\alpha}}{\|\boldsymbol{\alpha}\|^2} \diamond \boldsymbol{\mu} \right\|^2_{((\boldsymbol{\alpha}/\|\boldsymbol{\alpha}\|^2)\diamond\Sigma)^{-1}} \right)^{1/2} \|\mathbf{v}\|_{((\boldsymbol{\alpha}/\|\boldsymbol{\alpha}\|^2)\diamond\Sigma)^{-1}} \\
&= (2 + \|\boldsymbol{\alpha} \diamond \boldsymbol{\mu}\|^2_{(\boldsymbol{\alpha}\diamond\Sigma)^{-1}})^{1/2} \|\mathbf{v}\|_{(\boldsymbol{\alpha}\diamond\Sigma)^{-1}},
\end{aligned}
$$

$$\mathfrak{D}_0(\mathbf{v}) = \|\mathbf{v}\|^n_{((\boldsymbol{\alpha}/\|\boldsymbol{\alpha}\|^2)\diamond\Sigma)^{-1}} \left|\frac{\boldsymbol{\alpha}}{\|\boldsymbol{\alpha}\|^2} \diamond \Sigma\right|^{1/2} = \|\mathbf{v}\|^n_{(\boldsymbol{\alpha}\diamond\Sigma)^{-1}} |\boldsymbol{\alpha} \diamond \Sigma|^{1/2},$$

$$\mathfrak{E}_0(\mathbf{v}) = \left\langle \mathbf{v}, \frac{\boldsymbol{\alpha}}{\|\boldsymbol{\alpha}\|^2} \diamond \boldsymbol{\mu} \right\rangle_{((\boldsymbol{\alpha}/\|\boldsymbol{\alpha}\|^2)\diamond\Sigma)^{-1}} = \langle \mathbf{v}, \boldsymbol{\alpha} \diamond \boldsymbol{\mu}\rangle_{(\boldsymbol{\alpha}\diamond\Sigma)^{-1}}.$$

Thus, $f_0(\mathbf{v})$ matches the RHS of (4.2.5). Likewise, using (A.1.2), for $1 \leq k \leq n$, we have

$$f_k(v) := \beta_k \nu_{\{k\}}\left(v\mathbf{e}_k, \frac{\mathbf{e}_k}{\alpha_k}\right) = \left(\frac{2}{\pi}\right)^{1/2} \beta_k \frac{\exp(\mathfrak{E}_k(v))}{\mathfrak{D}_k(v)} \mathfrak{K}_{1/2}(\mathfrak{A}_k(v)), \quad v \in \mathbb{R}_*,$$

where

$$\mathfrak{A}_k(v) = \frac{|v|(2\Sigma_{kk} + \alpha_k\mu_k^2)^{1/2}}{\alpha_k^{1/2}\Sigma_{kk}}, \quad \mathfrak{D}_k(v) = |v|, \quad \mathfrak{E}_k(v) = \frac{v\mu_k}{\Sigma_{kk}}.$$

Thus, we find that $f_k(v)$ matches the RHS of (4.2.5). Hence, the Lévy measure $\mathcal{Y}$ satisfies (4.2.4). □

An alternative proof of the *WVAG* Lévy measure formula in (4.2.2) can be obtained using (3.2.3). Assume the setup in Remark 4.2.1. This proof has the advantage that it obtains the Lévy measure $\mathcal{Z}$ in (4.2.7) of the joint process $\mathbf{Z}$, which can be used to explain how $\mathbf{Y} \sim WVAG^n(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma)$ jumps based on the jumps of its subordinator $\mathbf{T} \sim AG^n(a, \boldsymbol{\alpha})$.

Let $G_0, \ldots, G_n$ be defined as in Definition 4.1.1. Recall the definition of C_J and V_J from Remark 3.2.5. Let J be the set of all indices $j \in \{1, \ldots, n\}$ such that B_j is a nonzero process, and assume that $J \neq \emptyset$.

In (4.2.7), $\mathcal{Z}(\{\mathbf{0}\} \times \mathbb{R}^n_*) = 0$, meaning that if $\mathbf{T}$ does not jump, then $\mathbf{Y}$ does not jump. Also, $\mathcal{Z}(C_{\{1,\ldots,n\}} \times V_J) = \infty$ and $\mathcal{Z}(C_{\{1,\ldots,n\}} \times (\mathbb{R}^n \backslash V_J)) = 0$, meaning that if a jump occurs in G_0, then a jump occurs in all components of $\mathbf{Y}$ where $\mathbf{B}$ is nonzero in the respective component. Thus, the subordinator G_0 models a common time change.

For $1 \leq k \leq n$, if $k \in J$, then $\mathcal{Z}(C_{\{k\}} \times V_{\{k\}}) = \infty$ and $\mathcal{Z}(C_{\{k\}} \times (\mathbb{R}^n \backslash V_{\{k\}})) = 0$, while if $k \notin J$, then $\mathcal{Z}(C_{\{k\}} \times \{\mathbf{0}\}) = \infty$ and $\mathcal{Z}(C_{\{k\}} \times \mathbb{R}^n_*) = 0$, meaning that if a jump occurs in G_k, then a jump occurs in Y_k, unless B_k is a zero process, in which case, Y_k obviously cannot jump. Thus, the subordinators $G_1, \ldots, G_n$ model idiosyncratic time changes.

Note that the above cases exhaust all possible jump sizes of $\mathbf{T}$. In summary, the jumps of the *AG* subordinator are inherited by the *WVAG* process whenever possible. This resembles the jump behaviour of strongly subordinated processes.

If Σ is a diagonal matrix, the characteristics in Proposition 4.2.2 reduce to those of a VAG process as specified in Theorem 1.1 of [LS10].

4.3 Properties

In this section, earlier results are applied to the $WVAG$ processes to prove some useful properties. Firstly, we show that their marginal components are VG processes, and in fact, these are the same marginal components as that of the corresponding VAG process.

Proposition 4.3.1. *If* $\mathbf{Y} \sim WVAG^n(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma)$*, then* $Y_k \sim VG^1(1/\alpha_k, \mu_k, \Sigma_{kk})$*,* $1 \le k \le n$*.*

Proof. Assume that $\mathbf{B} \sim BM^n(\boldsymbol{\mu}, \Sigma)$ and $\mathbf{T} \sim AG^n(a, \boldsymbol{\alpha})$ are independent. Thus, $B_k \sim BM^1(\mu_k, \Sigma_{kk})$ and $T_k \sim \Gamma_S(1/\alpha_k)$, the latter by (4.1.3). Now $\mathbf{Y} \stackrel{D}{=} \mathbf{B} \odot \mathbf{T}$, and applying Corollary 2.3.7, we have $Y_k \stackrel{D}{=} B_k \circ T_k \sim VG^1(1/\alpha_k, \mu_k, \Sigma_{kk})$, $1 \le k \le n$. □

Next, we show that the sample paths of a $WVAG$ process are of finite variation.

Proposition 4.3.2. *If* $\mathbf{Y} \sim WVAG^n(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma)$*, then* $\mathbf{Y} \sim FV^n$ *and driftless.*

Proof. By Proposition 4.2.2 (ii), $\mathbf{Y} \sim VGG^n(\mathbf{0}, \boldsymbol{\mu}, \Sigma, \mathcal{U}_{a,\boldsymbol{\alpha}})$. The finitely supported Thorin measure $\mathcal{U}_{a,\boldsymbol{\alpha}}$ trivially satisfies $\int_{\mathbb{D}^C} \mathcal{U}_{a,\boldsymbol{\alpha}}(\mathrm{d}\mathbf{u})/\|\mathbf{u}\|^{1/2} < \infty$, so the result follows from Proposition 3.3.1 (iii). □

Now we give moment formulas for $WVAG$ processes.

Proposition 4.3.3. *Let* $\mathbf{T}$ *and* $\mathbf{Y}$ *be defined as in Remark 4.2.1. For* $t > 0$*,*

$$\frac{\mathbb{E}[\mathbf{T}(t)]}{t} = \mathbf{e}, \tag{4.3.1}$$

$$\frac{\mathrm{Cov}(T_k(t), T_l(t))}{t} = \begin{cases} \alpha_k & \text{if } k = l, \\ a\alpha_k\alpha_l & \text{if } k \neq l, \end{cases} \tag{4.3.2}$$

$$\frac{\mathbb{E}[\mathbf{Y}(t)]}{t} = \boldsymbol{\mu}, \tag{4.3.3}$$

$$\frac{\mathrm{Cov}(Y_k(t), Y_l(t))}{t} = \begin{cases} \Sigma_{kk} + \alpha_k\mu_k^2 & \text{if } k = l, \\ a(\alpha_k \wedge \alpha_l)\Sigma_{kl} + a\alpha_k\alpha_l\mu_k\mu_l & \text{if } k \neq l, \end{cases} \tag{4.3.4}$$

$$\frac{\mathrm{Cov}(T_k(t), Y_l(t))}{t} = \begin{cases} \alpha_k\mu_l & \text{if } k = l, \\ a\alpha_k\alpha_l\mu_l & \text{if } k \neq l. \end{cases} \tag{4.3.5}$$

Proof. Since the expected value and covariance of a Lévy process are linear functions in t (see Proposition 1.1.9), it is enough to prove the results for $t = 1$. Since $T_k \sim \Gamma_S(1/\alpha_k)$ by (4.1.3), we have $\mathbb{E}[T_k(1)] = 1$, $\operatorname{Var}(T_k(1)) = \alpha_k$, $1 \le k \le n$. By (4.1.2),

$$\operatorname{Cov}(T_k(1), T_l(1)) = \alpha_k \alpha_l \operatorname{Var}(G_0(1)) = a\alpha_k\alpha_l, \quad 1 \le k \ne l \le n. \tag{4.3.6}$$

So we have proved (4.3.1) and (4.3.2).

We can apply the results of Example 2.3.24 to $\mathbf{Y} \overset{D}{=} \mathbf{B} \odot \mathbf{T}$. By (2.3.20) and (2.3.21), we have

$$\mathbb{E}[Y_k(1)] = \mu_k, \quad \operatorname{Var}(Y_k(1)) = \Sigma_{kk} + \alpha_k\mu_k^2, \quad 1 \le k \le n. \tag{4.3.7}$$

Let $1 \le k \ne l \le n$. By (2.3.22) and (4.1.6), we have

$$\int_{(0,\infty)} \tau_{k,l}(u)\,\mathrm{d}u = \int_{(0,\infty)} (g\alpha_k) \wedge (g\alpha_l)\,\mathcal{G}_{a,1}(\mathrm{d}g) = a(\alpha_k \wedge \alpha_l).$$

Substituting this and (4.3.6) into (2.3.23) gives the covariance formula

$$\operatorname{Cov}(Y_k(1), Y_l(1)) = a(\alpha_k \wedge \alpha_l)\Sigma_{kl} + a\alpha_k\alpha_l\mu_k\mu_l. \tag{4.3.8}$$

So we have proved (4.3.3) and (4.3.4).

Let $\mathbf{t} = (t_1, \ldots, t_n)$ and $1 \le k, l \le n$. By (2.3.19) and (2.3.16), we have

$$\operatorname{Cov}(T_k(1), Y_l(1)) = \int_{[0,\infty)^n_*} t_k \mathbb{E}[B_l(t_l)]\,\mathcal{T}_{a,\boldsymbol{\alpha}}(\mathrm{d}\mathbf{t}) = \mu_l \operatorname{Cov}(T_k(1), T_l(1)).$$

Thus, we obtain (4.3.5) by using (4.3.2). □

The moments of the AG subordinator and the VAG process given in Proposition 4.3.3 match those in [LS10, Sem08]. Due to Proposition 4.3.1, (4.3.7) obviously matches the moments of a VG^1 process (see page 85 in [MCC98]). The moments in Proposition 4.3.3 as well as higher order moments are listed in Appendix A.1 of [MS18].

For a VAG process, Σ is diagonal, so (4.3.4) implies $\operatorname{Cov}(Y_k(1), Y_l(1)) = a\alpha_k\alpha_l\mu_k\mu_l$ for $1 \le k \ne l \le n$. This gives rise to a restrictive dependence structure, a disadvantage that has been noted in [LS10]. In contrast, a $WVAG$ process has additional flexibility due to having the additional term $a(\alpha_k \wedge \alpha_l)\Sigma_{kl}$ when using a Brownian motion with dependent components as the subordinate.

Finally, we show that scaling the time parameter of a $WVAG$ process results in a $WVAG$ process with modified parameters. This property has been noted in Equation (20) of [MS18].

Proposition 4.3.4. *Let* $\mathbf{Y} \sim WVAG^n(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma)$ *and* $c > 0$. *Then* $(\mathbf{Y}(ct))_{t\geq 0} \sim WVAG^n(ca, \boldsymbol{\alpha}/c, c\boldsymbol{\mu}, c\Sigma)$.

Proof. Note that $(\mathbf{Y}(ct))_{t\geq 0} = \mathbf{Y} \circ (cI\mathbf{e})$, where I is the identity function, so it is a Lévy process by Proposition 1.3.2. Also, by (4.2.3),

$$
\begin{aligned}
-ac\ln\Big(1 - \mathrm{i}\Big\langle \frac{\boldsymbol{\alpha}}{c} \diamond (c\boldsymbol{\mu}), \boldsymbol{\theta}\Big\rangle + \frac{1}{2}\|\boldsymbol{\theta}\|^2_{(\boldsymbol{\alpha}/c)\diamond(c\Sigma)}\Big) \\
- \sum_{k=1}^{n} \frac{1 - a\alpha_k}{\alpha_k/c} \ln\Big(1 - \mathrm{i}\alpha_k\mu_k\theta_k + \frac{1}{2}\alpha_k\Sigma_{kk}\theta_k^2\Big)
\end{aligned}
$$

is equal to both the characteristic exponents of $(\mathbf{Y}(ct))_{t\geq 0}$ and the Lévy process with law $WVAG^n(ca, \boldsymbol{\alpha}/c, c\boldsymbol{\mu}, c\Sigma)$. □

4.4 Decomposition into Variance-Gamma Processes

Since an AG subordinator $\mathbf{T} \stackrel{D}{=} G_0\boldsymbol{\alpha} + (G_1, \dots, G_n) \sim AG^n(a, \boldsymbol{\alpha})$ is a superposition of independent ray subordinators, we can use the results in Section 2.3.4 to show that the $WVAG$ process can be written as a superposition of independent VG processes.

Proposition 4.4.1. *Let* $\mathbf{Z}$ *be defined as in Remark 4.2.1. Then*

$$
\mathbf{Z} \stackrel{D}{=} (G_0\boldsymbol{\alpha}, \mathbf{V}_0) + \sum_{k=1}^{n}(G_k\mathbf{e}_k, V_k\mathbf{e}_k), \tag{4.4.1}
$$

where

$$
\begin{aligned}
\mathbf{V}_0 &:= \mathbf{B}^{(\boldsymbol{\alpha})} \circ (G_0\mathbf{e}) \sim VG^n(a, a\boldsymbol{\alpha} \diamond \boldsymbol{\mu}, a\boldsymbol{\alpha} \diamond \Sigma), \\
V_k &:= B_k^* \circ G_k \sim VG^1(\beta_k, (1 - a\alpha_k)\mu_k, (1 - a\alpha_k)\Sigma_{kk}), \quad 1 \leq k \leq n,
\end{aligned}
$$

$\mathbf{B}^{(\boldsymbol{\alpha})} \sim BM^n(\boldsymbol{\alpha} \diamond \boldsymbol{\mu}, \boldsymbol{\alpha} \diamond \Sigma)$, $\mathbf{B}^* = (B_1^*, \dots, B_n^*) \sim BM^n(\boldsymbol{\mu}, \mathrm{diag}(\Sigma_{11}, \dots, \Sigma_{nn}))$, $G_0, \dots, G_n$ *are defined in Definition 4.1.1 and* $\mathbf{B}^{(\boldsymbol{\alpha})}, \mathbf{B}^*, G_0, \dots, G_n$ *are independent.*

Proof. Recall that $\mathbf{Z} \stackrel{D}{=} (\mathbf{T}, \mathbf{B} \odot \mathbf{T})$ from Remark 4.2.1, and that $\mathbf{T} \stackrel{D}{=} G_0\boldsymbol{\alpha} + (G_1, \dots, G_n)$ by (4.1.2). Let $\mathbf{B}^{(0)}, \dots, \mathbf{B}^{(n)}, G_0, \dots, G_n$ be independent, where $\mathbf{B}^{(k)} \sim BM^n(\boldsymbol{\mu}, \Sigma)$, $0 \leq k \leq n$. Since $\mathbf{T}$ is a superposition of independent ray subordinators, using Proposition 2.3.15 and then Proposition 2.3.13, we have

$$
\begin{aligned}
\mathbf{Z} &\stackrel{D}{=} \left(G_0\boldsymbol{\alpha} + \sum_{k=1}^{n} G_k \mathbf{e}_k, \mathbf{B} \odot \left(G_0\boldsymbol{\alpha} + \sum_{k=1}^{n} G_k \mathbf{e}_k \right) \right) \\
&\stackrel{D}{=} (G_0\boldsymbol{\alpha}, \mathbf{B}^{(0)} \odot (G_0\boldsymbol{\alpha})) + \sum_{k=1}^{n} (G_k\mathbf{e}_k, \mathbf{B}^{(k)} \odot (G_k\mathbf{e}_k)) \\
&\stackrel{D}{=} (G_0\boldsymbol{\alpha}, \mathbf{B}^{(\boldsymbol{\alpha})} \circ (G_0\mathbf{e})) + \sum_{k=1}^{n} (G_k\mathbf{e}_k, \mathbf{B}^{(\mathbf{e}_k)} \circ (G_k\mathbf{e})), \qquad (4.4.2)
\end{aligned}
$$

where $\mathbf{B}^{(\boldsymbol{\alpha})} \sim BM^n(\boldsymbol{\alpha}\diamond\boldsymbol{\mu}, \boldsymbol{\alpha}\diamond\Sigma)$ and $\mathbf{B}^{(\mathbf{e}_k)} = (B_1^{(\mathbf{e}_k)}, \ldots, B_n^{(\mathbf{e}_k)}) \sim BM^n(\mathbf{e}_k\diamond\boldsymbol{\mu}, \mathbf{e}_k\diamond\Sigma)$, $1 \le k \le n$, are independent (see Example 1.2.3) and also independent of $G_0, \ldots, G_n$.

Note that $G_0/a \sim \Gamma_S(a)$, so $\mathbf{V}_0 := \mathbf{B}^{(\boldsymbol{\alpha})} \circ (G_0\mathbf{e}) \sim VG^n(a, a\boldsymbol{\alpha} \diamond \boldsymbol{\mu}, a\boldsymbol{\alpha} \diamond \Sigma)$. Let $\mathbf{B}^* = (B_1^*, \ldots, B_n^*) := (B_1^{(\mathbf{e}_1)}, \ldots, B_n^{(\mathbf{e}_n)}) \sim BM^n(\boldsymbol{\mu}, \mathrm{diag}(\Sigma_{11}, \ldots, \Sigma_{nn}))$. For $1 \le k \le n$, $\mathbf{B}^{(\mathbf{e}_k)} \circ (G_k\mathbf{e}) = (B_k^* \circ G_k)\mathbf{e}_k$ since all its components, except possibly the kth component, are zero. Note that $G_k/(1 - a\alpha_k) \sim \Gamma_S(\beta_k)$, so $V_k := B_k^* \circ G_k \sim VG^1(\beta_k, (1 - a\alpha_k)\mu_k, (1 - a\alpha_k)\Sigma_{kk})$. Therefore, (4.4.2) is the same as (4.4.1), completing the proof. □

Corollary 4.4.2. *If* $\mathbf{Y} \sim WVAG^n(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma)$, *then*

$$\mathbf{Y} \stackrel{D}{=} V_0 + (V_1, \ldots, V_n),$$

where $V_0 \sim VG^n(a, a\boldsymbol{\alpha} \diamond \boldsymbol{\mu}, a\boldsymbol{\alpha} \diamond \Sigma)$, $V_k \sim VG^1(\beta_k, (1 - a\alpha_k)\mu_k, (1 - a\alpha_k)\Sigma_{kk})$, $1 \le k \le n$, *are independent.*

Proof. This immediately follows from Proposition 4.4.1 upon taking the last n components of (4.4.1). □

This result gives an alternative derivation of the characteristic exponent of a $WVAG$ process and can be used to simulate a $WVAG$ process as explained in Section 4.6.1.

Remark 4.4.3. By Corollary 4.4.2, it turns out that a $WVAG^n$ process is a superposition of an independent VG^n process and independent VG^1 processes in the kth component, $1 \le k \le n$. Other multivariate extensions of VG processes have been constructed by explicitly superpositioning an independent VG^n process and independent VG^1 processes in the kth component (see [LMS16, Wan09]). In the former, these are called factor-based subordinated Brownian motions, and it is noted in [MS18] that they are linear transformations of weakly subordinated processes.

4.5 Fourier Invertibility

In this section, we derive a condition for the Fourier invertibility of $WVAG$ processes.

Recall that I is the identity function. Let $\mathbf{Y} \sim WVAG^n(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma)$, and recall that $\Phi_{\mathbf{Y}(t)}(\boldsymbol{\theta}) = \exp(t\Psi_{\mathbf{Y}}(\boldsymbol{\theta}))$, $\boldsymbol{\theta} \in \mathbb{R}^n$, $t \geq 0$, with $\Psi_{\mathbf{Y}}$ given in (4.2.3).

Proposition 4.5.1. *Let* $\mathbf{R} \stackrel{D}{=} I\boldsymbol{\eta} + \mathbf{Y}$, $\boldsymbol{\eta} \in \mathbb{R}^n$ *and* $\mathbf{Y} \sim WVAG^n(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma)$. *Let* $t \geq 0$. *If* $\Phi_{\mathbf{Y}(t)} \in L^1$, *then the density function of* $\mathbf{R}(t)$ *is*

$$f_{\mathbf{R}(t)}(\mathbf{r}) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-\mathrm{i}\langle \boldsymbol{\theta}, \mathbf{r} - \boldsymbol{\eta} \rangle} \Phi_{\mathbf{Y}(t)}(\boldsymbol{\theta})\, \mathrm{d}\boldsymbol{\theta}, \quad \mathbf{r} \in \mathbb{R}^n. \tag{4.5.1}$$

Proof. See Theorem 1.3.6 in [Sas13]. □

If $\Phi_{\mathbf{Y}(t)} \in L^1$, we say that $\mathbf{Y}(t)$ is *Fourier invertible* and we give a condition for this in terms of an inequality relating the parameters.

Writing $\Sigma = (\Sigma_{kl})$, if we assume that $\Sigma_{kk} > 0$ for all $1 \leq k \leq n$ or the stronger condition that Σ is invertible (see Lemma A.2.2 (iii)), then the density function of $\mathbf{R}(t)$, which is needed for maximum likelihood, exists for all $t > 0$. To see this, note that in Corollary 4.4.2, $(1 - a\alpha_k)\Sigma_{kk} > 0$ implies that $V_k(t)$, $1 \leq k \leq n$, is absolutely continuous with some density function $f_{V_k(t)}(v)$ (see Equation (2.10) in [BKMS17]). Hence, $(V_1(t), \ldots, V_n(t))$ is also absolutely continuous since it has density function $\prod_{k=1}^n f_{V_k(t)}(v_k)$. Thus, $\mathbf{R}(t)$ must be absolutely continuous with respect to the Lebesgue measure because it is a convolution with at least one absolutely continuous random vector $(V_1(t), \ldots, V_n(t))$ (see Appendix F, Equation (2) in [Sas13]). However, the density function of $\mathbf{R}(t)$ is not explicitly known, so it is computed using Fourier inversion, that is via Proposition 4.5.1.

The next lemma allows us to assume $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\eta} = \mathbf{0}$ when proving the Fourier invertibility.

Lemma 4.5.2. *Let* $\mathbf{B} \sim BM^n(\boldsymbol{\mu}, \Sigma)$, $\mathbf{B}^* \sim BM^n(\mathbf{0}, \Sigma)$, $\mathbf{T} \sim S^n(\mathbf{0}, \mathcal{T})$, $\mathbf{Y} \stackrel{D}{=} \mathbf{B} \odot \mathbf{T}$, $\mathbf{Y}^* \stackrel{D}{=} \mathbf{B}^* \odot \mathbf{T}$, $\mathbf{R} \stackrel{D}{=} I\boldsymbol{\eta} + \mathbf{Y}$, $\boldsymbol{\eta} \in \mathbb{R}^n$. *For all* $t \geq 0$ *and* $p > 0$, *if* $\Phi_{\mathbf{Y}^*(t)} \in L^p$, *then* $\Phi_{\mathbf{R}(t)} \in L^p$.

Proof. For all $t \geq 0$, $\Phi_{\mathbf{R}(t)}(\boldsymbol{\theta}) = e^{\mathrm{i}t\langle \boldsymbol{\theta}, \boldsymbol{\eta} \rangle} \Phi_{\mathbf{Y}(t)}(\boldsymbol{\theta})$, $\boldsymbol{\theta} \in \mathbb{R}^n$, so that $|\Phi_{\mathbf{R}(t)}(\boldsymbol{\theta})| = \exp(t\Re\Psi_{\mathbf{Y}}(\boldsymbol{\theta}))$. Using (2.3.1) and then Example 1.2.3, we have

$$\begin{aligned}
\Re\Psi_{\mathbf{Y}}(\boldsymbol{\theta}) &= \int_{[0,\infty)^n_*} (\Re\Phi_{\mathbf{B}(\mathbf{t})}(\boldsymbol{\theta}) - 1)\, \mathcal{T}(\mathrm{d}\mathbf{t}) \\
&\leq \int_{[0,\infty)^n_*} (|\Phi_{\mathbf{B}(\mathbf{t})}(\boldsymbol{\theta})| - 1)\, \mathcal{T}(\mathrm{d}\mathbf{t})
\end{aligned}$$

$$
\begin{aligned}
&= \int_{[0,\infty)^n_*} \left(\exp\left(-\frac{1}{2} \|\boldsymbol{\theta}\|^2_{\mathbf{t}\diamond\Sigma} \right) - 1 \right) \mathcal{T}(\mathrm{d}\mathbf{t}) \\
&= \Re\Psi_{\mathbf{Y}^*}(\boldsymbol{\theta}).
\end{aligned}
$$

Therefore, $|\Phi_{\mathbf{R}(t)}(\boldsymbol{\theta})| \le |\Phi_{\mathbf{Y}^*(t)}(\boldsymbol{\theta})|$, from which the result follows. □

We give a Fourier invertibility condition for VG processes.

Lemma 4.5.3. *Let* $\mathbf{V} \sim VG^n(b, \boldsymbol{\mu}, \Sigma)$*. Assume that* Σ *is invertible. Let* $p > 0$*. If* $pb > n/2$*, then* $\Phi_{\mathbf{V}} \in L^p$*.*

Proof. By Lemma 4.5.2, we can assume $\boldsymbol{\mu} = \mathbf{0}$. By (1.3.4), $\mathbf{V} \sim VG^n(b, 0, \Sigma)$ has characteristic function

$$
\Phi_{\mathbf{V}}(\boldsymbol{\theta}) = \left(1 + \frac{\|\boldsymbol{\theta}\|^2_\Sigma}{2b} \right)^{-b}, \quad \boldsymbol{\theta} \in \mathbb{R}^n.
$$

Using the Cholesky decomposition, $\Sigma = U'U$, where U is a triangular matrix with positive elements on the diagonal. Let $p > 0$. Making the transformation $\boldsymbol{\theta} = (2b)^{1/2}\mathbf{x}(U')^{-1}$, noting that $(U')^{-1}$ exists so that the transformation is injective, we have

$$
\int_{\mathbb{R}^n} |\Phi_{\mathbf{V}}(\boldsymbol{\theta})|^p \,\mathrm{d}\boldsymbol{\theta} = |(2b)^{1/2}U^{-1}| \int_{\mathbb{R}^n} \left(1 + \|\mathbf{x}\|^2\right)^{-pb} \mathrm{d}\mathbf{x}. \tag{4.5.2}
$$

Using the polar decomposition (see Lemma A.3.3) on the RHS of (4.5.2), we have that $\Phi_{\mathbf{V}} \in L^p$ if and only if

$$
\int_0^\infty (1 + r^2)^{-pb} r^{n-1} \,\mathrm{d}r < \infty,
$$

which is equivalent to $pb > n/2$. □

Recall the definition of β_k in (4.1.1). We now present the Fourier invertibility condition.

Proposition 4.5.4. *Let* $\mathbf{Y} \sim WVAG^n(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma)$ *and* $\mathbf{R} \stackrel{D}{=} I\boldsymbol{\eta} + \mathbf{Y}$*,* $\boldsymbol{\eta} \in \mathbb{R}^n$*. Assume that* Σ *is invertible. For* $t > 0$*, if*

$$
\left(\frac{a}{n} + \min_{1 \le k \le n} \beta_k \right) t > \frac{1}{2}, \tag{4.5.3}
$$

then $\Phi_{\mathbf{Y}(t)}, \Phi_{\mathbf{R}(t)} \in L^1$*.*

Proof. By Proposition 4.3.4, it suffices to prove the result for $t = 1$, and by Lemma 4.5.2, we can assume $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\eta} = \mathbf{0}$, so that $\mathbf{R} \sim WVAG^n(a, \boldsymbol{\alpha}, \mathbf{0}, \Sigma)$.

Let $\mathbf{V}_0 \sim VG^n(a, \mathbf{0}, a\boldsymbol{\alpha} \diamond \Sigma)$, $V_k \sim VG^1(\beta_k, 0, (1 - a\alpha_k)\Sigma_{kk})$, $1 \le k \le n$, be independent, and let $\mathbf{V}^* := (V_1, \ldots, V_n)$. By Corollary 4.4.2, $\mathbf{R}$ has characteristic function $\Phi_{\mathbf{R}}(\boldsymbol{\theta}) = \Phi_{\mathbf{V}_0}(\boldsymbol{\theta})\Phi_{\mathbf{V}^*}(\boldsymbol{\theta})$, where $\Phi_{\mathbf{V}^*}(\boldsymbol{\theta}) := \prod_{k=1}^n \Phi_{V_k}(\theta_k)$. For $p^{-1} + q^{-1} = 1$, $p, q > 1$, Hölder's inequality gives

$$\begin{aligned}
\int_{\mathbb{R}^n} |\Phi_{\mathbf{R}}(\boldsymbol{\theta})| \, \mathrm{d}\boldsymbol{\theta} &\le \left(\int_{\mathbb{R}^n} |\Phi_{\mathbf{V}_0}(\boldsymbol{\theta})|^p \, \mathrm{d}\boldsymbol{\theta} \right)^{1/p} \left(\int_{\mathbb{R}^n} |\Phi_{\mathbf{V}^*}(\boldsymbol{\theta})|^q \, \mathrm{d}\boldsymbol{\theta} \right)^{1/q} \\
&= \left(\int_{\mathbb{R}^n} |\Phi_{\mathbf{V}_0}(\boldsymbol{\theta})|^p \, \mathrm{d}\boldsymbol{\theta} \right)^{1/p} \prod_{k=1}^n \left(\int_{\mathbb{R}} |\Phi_{V_k}(\theta)|^q \, \mathrm{d}\theta \right)^{1/q}.
\end{aligned}$$

By Lemma 4.5.3, this integral is finite when $pa > n/2$, $q\beta_k > 1/2$ for all $1 \le k \le n$, with $p, q > 1$. Thus,

$$1 = \frac{1}{p} + \frac{1}{q} < 2\left(\frac{a}{n} \wedge \frac{1}{2} \right) + 2\left(\min_{1 \le k \le n} \beta_k \wedge \frac{1}{2} \right),$$

which is equivalent to (4.5.3). □

Remark 4.5.5. Let $V \sim VG^1(b, \mu, \Sigma)$. The condition for $V(1)$ to be Fourier invertible is identical to the condition for its density function having no singularity, which is $b > 1/2$ (see Section 7 in [KT08]).

We see that for sufficiently small $t > 0$, (4.5.3) will not be satisfied. This means that using (4.5.1) to compute the density function may not be valid when attempting parameter estimation for a $WVAG$ process based on observations from a sufficiently small sampling interval.

4.6 Calibration

In this section, we discuss calibration methods for a bivariate $WVAG$ process using both simulated and financial data.

The following sets up the model and notation we use throughout this section. Let $n = 2$. Recall that I is the identity function. Let

$$\mathbf{Y} = (Y_1, Y_2) \sim WVAG^2(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma), \quad \mathbf{R} = (R_1, R_2) \stackrel{D}{=} I\boldsymbol{\eta} + \mathbf{Y}, \quad \boldsymbol{\eta} \in \mathbb{R}^2. \quad (4.6.1)$$

Let $a > 0$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in (0, 1/a)^2$, $\boldsymbol{\mu} = (\mu_1, \mu_2) \in \mathbb{R}^2$, $\Sigma = (\Sigma_{kl}) \in \mathbb{R}^{2\times 2}$ be a covariance matrix. Let (S_1, S_2) be a bivariate price process following an exponential Lévy model

$$S_k(t) = S_k(0)\exp(R_k(t)), \quad t \geq 0, \quad k = 1, 2. \tag{4.6.2}$$

For $m \in \mathbb{N}$ and $c > 0$, using the independent and stationary increment property of $\mathbf{R}$, the log returns are iid and given by

$$\mathbf{R}^{(j)} = (R_1^{(j)}, R_2^{(j)}) := \left(\ln \frac{S_1(jc)}{S_1((j-1)c)}, \ln \frac{S_2(jc)}{S_2((j-1)c)}\right) \stackrel{D}{=} \mathbf{R}(c), \quad j = 1, \ldots, m.$$

Let $\mathbf{r}^{(j)} = (r_1^{(j)}, r_2^{(j)})$ be the observed value of $\mathbf{R}^{(j)}$, $j = 1, \ldots, m$, so that $\mathbf{r}^{(1)}, \ldots, \mathbf{r}^{(m)}$ represents m equally spaced discrete observations of the log returns with sampling interval $c > 0$.

We call this the *WVAG model*. If $\Sigma_{12} = 0$, we called it the *VAG model* as $\mathbf{Y}$ reduces to a VAG process by Lemma 4.1.7.

The properties of $(\mathbf{R}(ct))_{t\geq 0}$ can be determined by noting that $(\mathbf{Y}(ct))_{t\geq 0}$ is a $WVAG$ process with modified parameters given in Proposition 4.3.4.

4.6.1 Simulation Method

Corollary 4.4.2 can be used to simulate the Lévy process $\mathbf{R}$ under the $WVAG$ model. To do this, it suffices to know how to simulate a random vector $\mathbf{V}(1) \sim VG^n(b, \boldsymbol{\mu}, \Sigma)$. By definition $\mathbf{V}(1) = \mathbf{B} \circ (G(1)\mathbf{e})$, where $\mathbf{B} \sim BM^n(\boldsymbol{\mu}, \Sigma)$ and $G \sim \Gamma_S(b)$ are independent. Thus,

$$\mathbf{V}(1) \,|\, G(1) \sim N(G(1)\boldsymbol{\mu}, G(1)\Sigma),$$

so that to simulate $\mathbf{V}(1)$, we first simulate a gamma random variable $G(1) \sim \Gamma(b)$, and then conditional on the value $G(1) = g$, we simulate a normal random vector $N(g\boldsymbol{\mu}, g\Sigma)$, which we take to be $\mathbf{V}(1)$.

Now we can simulate $\mathbf{Y}(c)$ as it is a sum of an independent VG^n random vector and VG^1 random variables as determined by Corollary 4.4.2 and Proposition 4.3.4. Hence, we can simulate $\mathbf{R}(c) \stackrel{D}{=} c\boldsymbol{\eta} + \mathbf{Y}(c)$. In addition, using the independent and stationary increment property of the Lévy process $\mathbf{R}$, we can simulate

$$\mathbf{R}(t) = \sum_{k=0}^{j} \mathbf{R}^{(k)}, \quad t \in [jc, (j+1)c), \quad j = 0, 1, \ldots, m,$$

where $\mathbf{R}^{(0)} \equiv \mathbf{0}$, $\mathbf{R}^{(j)} \stackrel{D}{=} \mathbf{R}(c)$, $j = 1, \ldots, m$, are independent.

For the sampling intervals $c = 1, 0.1$ and $m = 1000$ observations, we make 100 simulations of $\mathbf{R}$, and estimate the parameters from the observations $\mathbf{r}^{(1)}, \ldots, \mathbf{r}^{(m)}$

with true parameters

$$a = 1, \quad \boldsymbol{\alpha} = (0.8, 0.6), \quad \boldsymbol{\mu} = (0.1, -0.3), \quad \Sigma = \begin{pmatrix} 1 & 0.6 \\ 0.6 & 1.2 \end{pmatrix}, \quad \boldsymbol{\eta} = (-0.1, 0.3).$$

4.6.2 Calibration Methods

The 10 parameters $(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma, \boldsymbol{\eta})$ of the $WVAG$ model are estimated from the observations $\mathbf{r}^{(1)}, \ldots, \mathbf{r}^{(m)}$ using method of moments (MOM), which is quick and easy to implement, maximum likelihood (ML) from [MS18], which may be expected to be asymptotically optimal under the model, and a modification of digital moment estimation (DME) from [Mad15], which we expect to be more robust to model misspecification.

Method of moments. The initial values of μ_k, α_k, Σ_{kk}, η_k, $k = 1, 2$, are obtained by using least squares on the first four central moments

$$\mathbb{E}[R_k(c)], \quad \mathbb{E}[(R_k(c) - \mathbb{E}[R_k(c)])^p], \quad p = 2, 3, 4,$$

with the corresponding sample moments. The initial values of the joint parameters a, Σ_{12} are obtained by using least squares on the central comoments

$$\mathbb{E}[(R_1(c) - \mathbb{E}[R_1(c)])^p (R_2(c) - \mathbb{E}[R_2(c)])^p], \quad p = 1, 2,$$

with the corresponding sample moments. However, the $p = 1$ case is excluded when fitting the VAG model as there is one less parameter. Using these initial values, least squares is solved over all parameters. Note that this last step has no effect if the above moments can be matched exactly. Formulas for the lower order moments and the covariance are given in Proposition 4.3.3 while the others can be found in Appendix A.1 of [MS18].

Maximum likelihood estimation. The density function $f_{\mathbf{R}(c)}$ is not explicitly known so it is numerically computed using Fourier inversion by (4.5.1). The numerical optimisation needed to implement ML requires initial values. The first initial values can be obtained by MOM. Using the first initial values, ML is applied to the marginal component observations $r_k^{(1)}, \ldots, r_k^{(m)}$ to obtain the second initial values of μ_k, α_k, Σ_{kk}, η_k, $k = 1, 2$, and using these, to the bivariate observations $\mathbf{r}^{(1)}, \ldots, \mathbf{r}^{(m)}$ to obtain the second initial values of a, Σ_{12}. Finally, using the second initial values, ML is applied on all parameters. For the VAG model, we apply this method with the constraint $\Sigma_{12} = 0$.

Digital moment estimation. Let $k = 1, 2$. Let $\mathbf{q}$ be the vector of 10 equally spaced points from 0.05 to 0.95 and $\mathbb{P}_k$ be the empirical quantiles of the observations $r_k^{(1)}, \dots, r_k^{(m)}$ at the probabilities $\mathbf{q}$. Recall that $R_k \overset{D}{=} I\eta_k + Y_k$, $Y_k \sim VG^1(1/\alpha_k, \mu_k, \Sigma_{kk})$ by Proposition 4.3.1. Let

$$p_r(\mu_k, \alpha_k, \Sigma_{kk}, \eta_k) := \mathbb{P}(R_k(c) \le r), \quad r \in \mathbb{P}_k,$$

and let q_r be the corresponding empirical probability. The marginal parameters $\mu_k, \alpha_k, \Sigma_{kk}, \eta_k$ are estimated by minimising the error

$$e_k(\mu_k, \alpha_k, \Sigma_{kk}, \eta_k) := \sum_{r \in \mathbb{P}_k} (p_r(\mu_k, \alpha_k, \Sigma_{kk}, \eta_k) - q_r)^2.$$

With the estimated marginal parameters, let $\rho := \Sigma_{12}/(\Sigma_{11}\Sigma_{22})^{1/2}$,

$$p_{\mathbf{r}}(a, \rho) := \mathbb{P}(R_1(c) \le r_1, R_2(c) \le r_2) \quad \mathbf{r} = (r_1, r_2), \in \mathbb{P}_1 \times \mathbb{P}_2,$$

and let $q_{\mathbf{r}}$ be the corresponding empirical probability. Recall that $\mathbf{R} \overset{D}{=} I\boldsymbol{\eta} + \mathbf{Y}$, $\mathbf{Y} \sim WVAG^2(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma)$. Since directly calculating $p_{\mathbf{r}}(a, \rho)$ requires Fourier inversion and is computationally expensive, it is estimated by the empirical probability over 10000 simulations. The joint parameters a, ρ are estimated by minimising the LOESS smooth [CGS91] of the error

$$e_0(a, \rho) := \sum_{\mathbf{r} \in \mathbb{P}_1 \times \mathbb{P}_2} (p_{\mathbf{r}}(a, \rho) - q_{\mathbf{r}})^2.$$

The predictor variables for the LOESS smooth are 100 equally spaced points on the feasible set $(a, \rho) \in (0, (1/\alpha_1) \wedge (1/\alpha_2)) \times (-1, 1)$. For the VAG model, we apply the above method with the constraint $\rho = 0$.

4.6.3 Goodness of Fit Statistics

To assess the overall goodness of fit of each parameter estimation method, as opposed to assessing individual parameters, we consider 3 goodness of fit statistics, the negative log-likelihood $(-\ln L)$, a chi-squared (χ^2) statistic, and a Kolmogorov-Smirnov (KS) statistic.

To compute χ^2, we apply the Rosenblatt transform [Ros52] of the fitted distribution to the observations. The Rosenblatt transform $\mathfrak{T}$ of $\mathbf{R}(c)$ is

$$\mathfrak{T}(\mathbf{r}) = (F_{R_1(c)}(r_1), F_{R_2(c)\,|\,R_1(c)}(r_2\,|\,r_1)), \quad \mathbf{r} = (r_1, r_2) \in \mathbb{R}^2,$$

where $F_{R_1(c)}$ is the cumulative distribution function of $R_1(c)$ and $F_{R_2(c)\,|\,R_1(c)}$ is the conditional cumulative distribution function of $R_2(c)$ given $R_1(c)$. This transform has the property that if the observations $\mathbf{R}^{(j)} \stackrel{D}{=} \mathbf{R}(c)$, $j = 1, \ldots, m$, then $\mathfrak{T}(\mathbf{R}^{(j)})$ are uniformly distributed on $[0,1]^2$. Under the $WVAG$ model, $R_1 \stackrel{D}{=} I\eta_1 + Y_1$, $Y_1 \sim VG^1(1/\alpha_1, \mu_1, \Sigma_{11})$.

Since VG^1 distributions have a known density function (see Equation (2.10) in [BKMS17]), we can obtain $f_{R_1(c)}$, and hence $F_{R_1(c)}$ by integration. In addition,

$$F_{R_2(c)\,|\,R_1(c)}(r_2\,|\,r_1) = \int_{-\infty}^{r_2} \frac{f_{\mathbf{R}(c)}(r_1, u)}{f_{R_1(c)}(r_1)}\,\mathrm{d}u,$$

where $f_{\mathbf{R}(c)}$ was computed by Fourier inversion using (4.5.1). To determine $\mathfrak{T}$, both $F_{R_1(c)}$ and $F_{R_2(c)\,|\,R_1(c)}$ are computed using the fitted parameter estimates. Then the χ^2 statistic is

$$\chi^2 = \sum_{i=1}^{l} \frac{(O_i - E_i)^2}{E_i},$$

where $[0,1]^2$ is partitioned into are $l = 100$ equal sized cells, O_i is the number of transformed observations $\mathfrak{T}(\mathbf{r}^{(1)}), \ldots, \mathfrak{T}(\mathbf{r}^{(m)})$ in the ith cell, and $E_i = m/l$ is the expected number of the latter under the uniform distribution.

Since computing $-\ln L$ and χ^2 requires Fourier inversion, it may not be possible to compute these statistics accurately when the Fourier invertibility condition does not hold. Therefore, we also consider the 2-dimensional, two-sample Kolmogorov-Smirnov statistic introduced by Peacock in [Pea83], and computed using the method in [Xia17]. This is the statistic for testing the equality of the fitted distribution and the true distribution based on a sample from the respective distributions, and therefore does not require the density function $f_{\mathbf{R}(c)}$ or Fourier inversion. When applied to real data in Subsection 4.6.6, we take the average of the KS statistics computed from the observations and 100 samples from the fitted distribution. When applied to simulated data in Subsection 4.6.5, the KS statistic is computed from the observations and a sample from the fitted distribution. All 3 goodness of fit statistics were averaged over the 100 simulations.

4.6.4 Quantile Choice for DME

Different choices of quantiles for DME are possible. Let $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, $\mathbf{q}_4$ be the vectors of 10 equally spaced points from 0.05 to 0.95, 10 equally spaced points from 0.01 to 0.99, 10 equally spaced points from 0.1 to 0.9, 20 equally spaced points from 0.05 to

Parameter	True value	$\mathbf{q}_1$	$\mathbf{q}_2$	$\mathbf{q}_3$	$\mathbf{q}_4$
a	1	0.171	0.171	0.182	0.175
α_1	0.8	0.127	0.132	0.143	0.128
α_2	0.6	0.126	0.145	0.149	0.129
μ_1	0.1	0.062	0.066	0.066	0.062
μ_2	-0.3	0.121	0.271	0.229	0.188
Σ_{11}	1	0.084	0.083	0.093	0.084
Σ_{22}	1.2	0.113	0.166	0.147	0.123
Σ_{12}	0.6	0.154	0.182	0.172	0.150
η_1	-0.1	0.051	0.054	0.053	0.050
η_2	0.3	0.110	0.262	0.219	0.179
$-\ln L$		2791.674	2795.826	2794.374	2792.303
χ^2		93.848	97.292	96.728	95.078
KS		0.054	0.055	0.054	0.054

Table 4.1: RMSE using DME with quantiles $\mathbf{q}_1, \ldots, \mathbf{q}_4$ for the $WVAG$ model fitted to simulated data with $c = 1$.

0.95, respectively. For the sampling interval $c = 1$, Table 4.1 shows the goodness of fit for these four choices of quantiles. We find that $\mathbf{q} = \mathbf{q}_1$ yields the lowest RMSE for most variables and has the lowest goodness of fit statistics. However, given that the results are so similar, these quantile choices make only a small difference to the overall goodness of fit.

4.6.5 Simulated Data Results

For the sampling interval $c = 1$, the Fourier invertibility condition is satisfied as the LHS of (4.5.3) is $0.75 > 1/2$. The calibration results for the $WVAG$ model with $c = 1$ is shown in Table 4.2. Here, we find that ML gives the best fit with the lowest χ^2 statistic. The KS statistic for ML and DME are approximately equal.

For the sampling interval $c = 0.1$, the Fourier invertibility condition is violated as the LHS of (4.5.3) is $0.08 < 1/2$. The corresponding results are shown in Table 4.3. Note that the goodness of fit statistics $-\ln L$ and χ^2 are not displayed in Table 4.3 as they may be inaccurate due to requiring Fourier inversion to compute. Here, we find that DME gives the best fit with the lowest KS statistic, however ML still produces an accurate fit and does not break down. This suggests that the Fourier invertibility condition holding may not be a requirement for ML to produce accurate parameter estimates. In both cases, $c = 1, 0.1$, the RMSEs and goodness of fit statistics are highest for MOM.

Parameter	True value	MOM	ML	DME
a	1	0.920 (0.424)	0.983 (0.242)	0.902 (0.171)
α_1	0.8	0.806 (0.342)	0.824 (0.111)	0.818 (0.127)
α_2	0.6	0.589 (0.216)	0.594 (0.094)	0.589 (0.126)
μ_1	0.1	0.103 (0.097)	0.103 (0.053)	0.096 (0.062)
μ_2	−0.3	−0.310 (0.131)	−0.301 (0.083)	−0.313 (0.121)
Σ_{11}	1	0.989 (0.078)	1.006 (0.071)	0.993 (0.084)
Σ_{22}	1.2	1.177 (0.088)	1.202 (0.086)	1.179 (0.113)
Σ_{12}	0.6	0.835 (0.335)	0.669 (0.192)	0.639 (0.154)
η_1	−0.1	−0.103 (0.089)	−0.105 (0.045)	−0.097 (0.051)
η_2	0.3	0.313 (0.120)	0.302 (0.070)	0.314 (0.110)
$-\ln L$		2802.337	2787.513	2791.674
χ^2		119.052	91.268	93.848
KS		0.068	0.054	0.054

Table 4.2: Expected value of estimates and RMSE (in parentheses) for the $WVAG$ model fitted to simulated data with $c = 1$.

Parameter	True value	MOM	ML	DME
a	1	1.106 (0.507)	0.990 (0.062)	0.896 (0.121)
α_1	0.8	0.636 (0.247)	0.782 (0.033)	0.796 (0.057)
α_2	0.6	0.504 (0.198)	0.602 (0.026)	0.603 (0.031)
μ_1	0.1	0.099 (0.167)	0.114 (0.099)	0.104 (0.170)
μ_2	−0.3	−0.347 (0.219)	−0.250 (0.123)	−0.301 (0.146)
Σ_{11}	1	0.992 (0.136)	1.005 (0.133)	1.013 (0.302)
Σ_{22}	1.2	1.197 (0.166)	1.245 (0.161)	1.234 (0.221)
Σ_{12}	0.6	0.842 (0.353)	0.262 (0.364)	0.564 (0.188)
η_1	−0.1	−0.111 (0.128)	−0.114 (0.015)	−0.100 (0.000)
η_2	0.3	0.351 (0.164)	0.288 (0.014)	0.300 (0.001)
KS		0.326	0.222	0.078

Table 4.3: Expected value of estimates and RMSE (in parentheses) for the $WVAG$ model fitted to simulated data with $c = 0.1$.

4.6.6 Financial Data Results

Next, we fit the $WVAG$ and VAG models to the S&P500 and FTSE100 indices as the bivariate price process (S_1, S_2) in (4.6.2) for a five-year period from 14 February 2011 to 12 February 2016 with daily closing price observations taking $c = 1$. There are 1249 bivariate observations. The estimated parameters, goodness of fit statistics and standard errors, computed using 100 bootstrap samples, are listed in Table 4.4. Contour plots of the fitted distributions and scatter plots of the bivariate log returns are shown in Figure 4.1.

For the $WVAG$ and VAG models, the LHS of (4.5.3) using the fitted parameter

	MOM		ML		DME	
Parameter	*WVAG*	*VAG*	*WVAG*	*VAG*	*WVAG*	*VAG*
a	0.695 (0.199)	0.696 (0.217)	0.962 (0.139)	1.087 (0.117)	0.899 (0.138)	1.114 (0.327)
α_1	1.436 (0.398)	1.436 (0.349)	0.919 (0.105)	0.908 (0.104)	0.898 (0.143)	
α_2	0.933 (0.277)	0.707 (0.120)	0.837 (0.105)	0.919 (0.085)	0.878 (0.132)	
$1000\mu_1$	−1.153 (0.704)	−1.156 (0.595)	−0.403 (0.550)	−0.630 (0.694)	−0.562 (0.563)	
$1000\mu_2$	−1.246 (0.704)	−1.255 (0.789)	−0.891 (0.586)	−0.867 (0.578)	−1.166 (0.554)	
$10000\Sigma_{11}$	0.982 (0.070)	0.980 (0.069)	0.976 (0.061)	0.935 (0.064)	0.928 (0.071)	
$10000\Sigma_{22}$	1.006 (0.053)	1.007 (0.062)	1.028 (0.063)	1.014 (0.060)	1.051 (0.086)	
$10000\Sigma_{12}$	0.994 (0.072)	–	0.813 (0.089)	–	0.844 (0.095)	–
$1000\eta_1$	1.422 (0.604)	1.426 (0.519)	0.705 (0.430)	0.879 (0.575)	0.982 (0.444)	
$1000\eta_2$	1.198 (0.617)	1.207 (0.722)	0.847 (0.440)	0.882 (0.470)	1.066 (0.410)	
$-\ln L$	−8465.969	−8192.168	−8496.315	−8239.301	−8492.798	−8237.767
χ^2	144.034	702.241	118.574	586.789	99.359	584.547
KS	0.073	0.151	0.050	0.139	0.048	0.138

Table 4.4: Parameter estimates and standard errors (in parentheses) for the $WVAG$ and VAG models fitted to the S&P500-FTSE100 data set. For DME, estimates for the marginal parameters of the $WVAG$ and VAG models are identical.

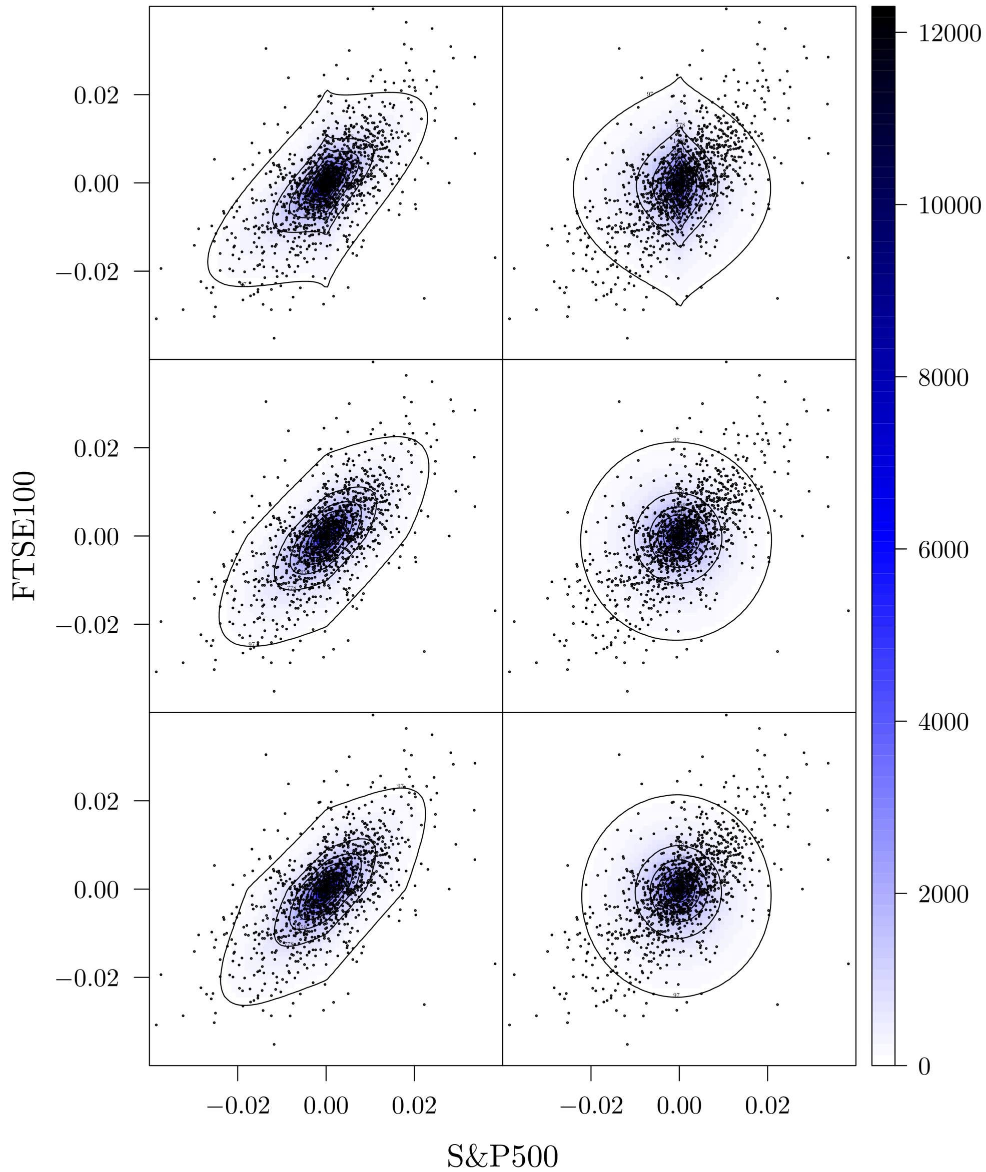

Figure 4.1: Scatterplots of log returns of the S&P500-FTSE100 data set and contour plots of the fitted distributions using the $WVAG$ model (left) and VAG model (right) with MOM (top), ML estimation (middle), DME (bottom). The contours in each plot, from outer to inner, correspond to the values $97+(k-1)681$, $k=1,\dots,15$, respectively. The display is restricted to $[-0.04,0.04]^2$ though several points are outside this region.

estimates are 0.61 and 0.54, respectively, so the Fourier invertibility condition is satisfied for both models. Based on the χ^2, KS statistic and contour plots, the $WVAG$ model produces a better fit than the VAG model. In addition, for the $WVAG$ model, DME gives a fit with a lower χ^2 and KS statistic than ML and MOM. This is consistent with the findings of [Mad11], though that is in the context of univariate calibration.

Assuming that the log returns satisfy the $WVAG$ model, a likelihood ratio test can be used to test the hypothesis $H_0 : \Sigma_{12} = 0$ versus $H_1 : \Sigma_{12} \neq 0$. The test statistic $D = 514.03$ is asymptotically χ^2 distributed with 1 degree of freedom. The p-value is less than 10^{-4}, so the VAG model is rejected. Indeed, the VAG model is not suited for modelling strong correlation since $\text{Cov}(Y_1(1), Y_2(1)) = a\alpha_1\alpha_2\mu_1\mu_2$ (4.3.4), which is approximately 0 when $\mu_1\mu_2$ is.

Chapter 5

Self-Decomposability of Weak Variance Generalised Gamma Convolutions

Self-decomposable processes are an important subclass of Lévy processes. The self-decomposability of $VGG^{n,1}$ processes has been widely studied, most recently by Grigelionis [Gri07b] who showed that if $n \geq 2$, then the Brownian motion subordinate being driftless implies self-decomposability, and under some moment conditions on the underlying Thorin measure, this is also necessary. Here, we extend this investigation to weak variance generalised gamma convolutions, providing necessary conditions as well as sufficient conditions for self-decomposability.

In Section 5.1, we give a brief introduction to self-decomposable processes. In Section 5.2, we give sufficient conditions for the self-decomposability of a weak variance generalised gamma convolution. In Section 5.3, we give necessary conditions. In Section 5.4, we develop several technical lemmas needed to prove the necessary conditions. In Section 5.5, we apply these results to various examples of VGG^n processes, including $VGG^{n,n}$ processes and $WVAG$ processes. In addition, we show that the moment conditions for necessity fail to be satisfied for some other VGG^n processes, including for generalised hyperbolic and $CGMY$ processes.

5.1 Self-Decomposable Processes

We begin with the definition of a self-decomposable process.

Definition 5.1.1. An n-dimensional random vector $\mathbf{X}$ is *self-decomposable* (SD) if for any $0 < b < 1$, there exists an n-dimensional random vector $\mathbf{Z}_b$ such that

$$\Phi_{\mathbf{X}}(\boldsymbol{\theta}) = \Phi_{\mathbf{Z}_b}(\boldsymbol{\theta})\Phi_{\mathbf{X}}(b\boldsymbol{\theta}), \quad \boldsymbol{\theta} \in \mathbb{R}^n. \tag{5.1.1}$$

An n-dimensional Lévy process $\mathbf{Y}$ is *self-decomposable* if $\mathbf{Y}(1)$ is.

In this definition, (5.1.1) means that $\mathbf{X} \stackrel{D}{=} b\mathbf{X} + \mathbf{Z}_b$ for some $\mathbf{Z}_b$ independent of $\mathbf{X}$. In addition, SD distributions are a subclass of infinitely divisible distributions and SD processes are a subclass of Lévy processes. We see in the next lemma that the class of SD processes is closed under convolution and convergence in distribution. Recall that I is the identity function.

Lemma 5.1.2. *For $m \geq 1$, if $\mathbf{Y}^{(1)}, \ldots, \mathbf{Y}^{(m)}$ are independent SD processes, then $\sum_{k=1}^{m} \mathbf{Y}^{(m)}$ is a SD process. If $\mathbf{Y}^{(k)}$, $k \in \mathbb{N}$, are SD processes and $\mathbf{Y}$ is a Lévy process such that $\mathbf{Y}^{(k)}(1) \stackrel{D}{\to} \mathbf{Y}(1)$ as $k \to \infty$, then $\mathbf{Y}$ is a SD process. If $\mathbf{Y}$ is a SD process, then $I\boldsymbol{\eta} + a\mathbf{Y}$, where $\boldsymbol{\eta} \in \mathbb{R}^n$ and $a > 0$, is a SD process.*

Proof. See Corollary 2.2 in [Sat80]. □

Recall that $\mathbb{S} := \{\mathbf{s} \in \mathbb{R}^n : \|\mathbf{s}\| = 1\}$. The next proposition is an important and commonly used characterisation of SD processes.

Lemma 5.1.3. *A Lévy process $\mathbf{Y} \sim L^n(\boldsymbol{\mu}, \Sigma, \mathcal{Y})$ is self-decomposible if and only if there exists a finite measure $\mathcal{S}$ on $\mathbb{S}$ and a nonnegative function $k(\mathbf{s}, r)$, $\mathbf{s} \in \mathbb{S}$, $r > 0$, where $\mathbf{s} \mapsto k(\mathbf{s}, r)$ is measurable for all $r > 0$ and $r \mapsto k(\mathbf{s}, r)$ is nonincreasing for all $\mathbf{s} \in \mathbb{S}$, such that*

$$\mathcal{Y}(A) = \int_{\mathbb{S}} \int_{(0,\infty)} \mathbf{1}_A(r\mathbf{s}) k(\mathbf{s}, r) \frac{\mathrm{d}r}{r} \mathcal{S}(\mathrm{d}\mathbf{s}) \tag{5.1.2}$$

for all Borel sets $A \subseteq \mathbb{R}^n_*$.

Proof. See Theorem 15.10 in [Sat99]. □

The following example shows that Thorin subordinators are a subclass of SD processes. Recall that $\mathbb{S}_+ := \mathbb{S} \cap [0, \infty)^n$.

Example 5.1.4. Let $\mathbf{T} \sim GGC^n_S(\mathbf{d}, \mathcal{T})$ be a Thorin subordinator. Its Lévy measure $\mathcal{T}$ is given in (3.1.4) in the required form (5.1.2). With $k(\mathbf{s}, r)$ given in (3.1.5), $\mathbf{s} \mapsto k(\mathbf{s}, r)$ is measurable for all $r > 0$ and it follows from Theorem 3.1.1 in [Bon92] that $r \mapsto k(\mathbf{s}, r)$ is nonincreasing for all $\mathbf{s} \in \mathbb{S}_+$. Thus, $\mathbf{T} \sim GGC^n_S(\mathbf{d}, \mathcal{T})$ is a SD process.

It turns out that $r \mapsto k(\mathbf{s}, r) = r(\mathrm{d}\mathcal{T}_{\mathbf{s}}/\mathrm{d}x)(r)$, where $\mathrm{d}\mathcal{T}_{\mathbf{s}}/\mathrm{d}x$ is the Lévy density of some univariate Thorin subordinator, and this function is completely monotone.

While all Thorin subordinators are SD, we now turn to the question of whether or not this continues to hold when they are weakly subordinated with Brownian motion, producing a VGG^n process. The self-decomposability of $VGG^{n,1}$ processes was investigated by Grigelionis [Gri07b] and we state this result below.

Lemma 5.1.5. *Let $\mathbf{Y} \sim VGG^{n,1}(d, \boldsymbol{\mu}, \Sigma, \mathcal{U})$.*

(i) If $n = 1$, or $n \geq 2$ and $\boldsymbol{\mu} = \mathbf{0}$, then $\mathbf{Y}$ is SD.

(ii) If $n = 2$, $|\Sigma| \neq 0$, $\boldsymbol{\mu} \neq \mathbf{0}$ and $0 < \int_{(0,\infty)} (1+u)^2 \mathcal{U}(\mathrm{d}u) < \infty$, or if $n \geq 3$, $|\Sigma| \neq 0$, $\boldsymbol{\mu} \neq \mathbf{0}$ and $0 < \int_{(0,\infty)} (1+u) \mathcal{U}(\mathrm{d}u) < \infty$, then $\mathbf{Y}$ is not SD.

Proof. See Proposition 3 in [Gri07b]. □

5.2 Sufficient Conditions

We now prove that VGG^n processes are SD when the Brownian motion subordinate is driftless. This is an analogous result to Lemma 5.1.5 (i).

Lemma 5.2.1. *If $\mathbf{Y}^* \sim VGG^n(\mathbf{0}, \mathbf{0}, \Sigma, \mathcal{U})$ is SD, then $\mathbf{Y} \sim VGG^n(\mathbf{d}, \mathbf{0}, \Sigma, \mathcal{U})$ is SD.*

Proof. For $\mathbf{Y} \sim VGG^n(\mathbf{d}, \mathbf{0}, \Sigma, \mathcal{U})$, we have $\mathbf{Y} \stackrel{D}{=} \mathbf{Y}_\emptyset + \mathbf{Y}^*$ by Remark 3.2.5. Here, $\mathbf{Y}_\emptyset \sim BM^n(\mathbf{0}, \mathbf{d} \diamond \Sigma)$ due to Proposition 2.3.13, and $\mathbf{Y}^* \sim VGG^n(\mathbf{0}, \mathbf{0}, \Sigma, \mathcal{U})$. In addition, $\mathbf{Y}_\emptyset$ and $\mathbf{Y}^*$ are independent. Now $\mathbf{Y}_\emptyset$, which has Lévy measure 0, is SD because the conditions in Lemma 5.1.3 are trivially satisfied, while $\mathbf{Y}^*$ is SD by assumption. Thus, $\mathbf{Y}$ is SD by Lemma 5.1.2. □

Theorem 5.2.2. *Let $\mathbf{Y} \sim VGG^n(\mathbf{d}, \boldsymbol{\mu}, \Sigma, \mathcal{U})$. If $n = 1$, or $n \geq 2$ and $\boldsymbol{\mu} = \mathbf{0}$, then $\mathbf{Y}$ is SD.*

Proof. If $n = 1$, then by Example 3.2.4, $Y \sim VGG^{1,1}(d, \mu, \Sigma, \mathcal{U})$, so the result follows from Lemma 5.1.5 (i). Now assume $n \geq 2$ and $\boldsymbol{\mu} = \mathbf{0}$. By Lemma 5.2.1, we also assume $\mathbf{d} = \mathbf{0}$ without loss of generality.

Finitely supported Thorin measure. Suppose $\mathbf{Y} \sim VGG^n(\mathbf{0}, \mathbf{0}, \Sigma, \mathcal{U})$, where $\mathcal{U} = \sum_{k=1}^m u_k \boldsymbol{\delta}_{\boldsymbol{\alpha}_k}$ for some $u_k \geq 0$, $\boldsymbol{\alpha}_k \in [0, \infty)^n_*$, $1 \leq k \leq m$, $m \geq 1$. By (3.2.4), we have

$$\Psi_{\mathbf{Y}}(\boldsymbol{\theta}) = -\sum_{k=1}^m u_k \ln\left(1 + \frac{\|\boldsymbol{\theta}\|^2_{\boldsymbol{\alpha}_k \diamond \Sigma}}{2\|\boldsymbol{\alpha}_k\|^2}\right), \quad \boldsymbol{\theta} \in \mathbb{R}^n.$$

Let $\mathbf{Y}^{(1)}, \ldots, \mathbf{Y}^{(m)}$ be independent with $\mathbf{Y}^{(k)} \sim VGG^{n,1}(0, \mathbf{0}, (\boldsymbol{\alpha}_k \diamond \Sigma)/w_k, u_k \boldsymbol{\delta}_{w_k})$, where $w_k := \|\boldsymbol{\alpha}_k\|/\|\mathbf{e}\|$, $1 \leq k \leq m$, so that $\mathbf{Y}^{(k)}$ is SD by Lemma 5.1.5 (i). Note that $\mathbf{Y}^{(k)} \sim VGG^n(\mathbf{0}, \mathbf{0}, (\boldsymbol{\alpha}_k \diamond \Sigma)/w_k, u_k\boldsymbol{\delta}_{w_k\mathbf{e}})$ by Example 3.2.4, so applying (3.2.4) yields

$$\begin{aligned}\Psi_{\mathbf{Y}^{(k)}}(\boldsymbol{\theta}) &= -u_k \ln\left(1 + \frac{\|\boldsymbol{\theta}\|^2_{(w_k\mathbf{e})\diamond((\boldsymbol{\alpha}_k\diamond\Sigma)/w_k)}}{2\|w_k\mathbf{e}\|^2}\right)\\ &= -u_k \ln\left(1 + \frac{\|\boldsymbol{\theta}\|^2_{\boldsymbol{\alpha}_k\diamond\Sigma}}{2\|\boldsymbol{\alpha}_k\|^2}\right), \quad \boldsymbol{\theta} \in \mathbb{R}^n,\end{aligned}$$

which implies $\Psi_{\mathbf{Y}} = \sum_{k=1}^m \Psi_{\mathbf{Y}^{(k)}}$. In particular, $\mathbf{Y} \overset{D}{=} \sum_{k=1}^m \mathbf{Y}^{(k)}$ is SD as a superposition of independent SD processes by Lemma 5.1.2.

Arbitrary Thorin measure. Let $\mathbf{Y} \sim VGG^n(\mathbf{0}, \mathbf{0}, \Sigma, \mathcal{U})$, where $\mathcal{U}$ is an arbitrary Thorin measure. Let

$$\begin{aligned}w(\mathbf{u}) &:= (1 + \ln^- \|\mathbf{u}\|) \wedge \|\mathbf{u}\|^{-1}, \quad \mathbf{u} \in [0,\infty)^n_*,\\ \widetilde{w}(r) &:= (1 + \ln^- r) \wedge r^{-1}, \quad r > 0,\end{aligned}$$

and $I := \int_{[0,\infty)^n_*} w(\mathbf{u})\mathcal{U}(\mathrm{d}\mathbf{u}) \in [0,\infty)$ from (3.1.1). If $I = 0$, then $\mathcal{U} = 0$, so $\mathbf{Y}$ is SD. Otherwise, if $I > 0$, let $\mathcal{U}^1(\mathrm{d}\mathbf{u}) := w(\mathbf{u})\mathcal{U}(\mathrm{d}\mathbf{u})/I$ be a Borel probability measure on $[0,\infty)^n_*$.

By Corollaries 30.5 and 30.9 in [Bau92], there exists a sequence of finitely supported Borel probability measures $(\mathcal{U}^1_k)_{k\in\mathbb{N}}$ on the locally compact space $[0,\infty)^n_*$ with infinitely remote point $\mathbf{0}$, such that $\mathcal{U}^1_k$ converges weakly to $\mathcal{U}^1$ as $k \to \infty$. For $k \in \mathbb{N}$, let $\mathcal{U}_k(\mathrm{d}\mathbf{u}) := I\mathcal{U}^1_k(\mathrm{d}\mathbf{u})/w(\mathbf{u})$. By construction, this is a finitely supported Thorin measure, so the associated Lévy process $\mathbf{Y}^{(k)} \sim VGG^n(\mathbf{0}, \mathbf{0}, \Sigma, \mathcal{U}_k)$ is self-decomposable by the first part of this proof.

Let $\boldsymbol{\theta} \in \mathbb{R}^n$, and define the nonnegative and continuous functions

$$\begin{aligned}\mathbf{u} \mapsto g_{\boldsymbol{\theta}}(\mathbf{u}) &:= \frac{1}{w(\mathbf{u})} \ln\left(1 + \frac{\|\boldsymbol{\theta}\|^2_{\mathbf{u}\diamond\Sigma}}{2\|\mathbf{u}\|^2}\right), \quad \mathbf{u} \in [0,\infty)^n_*,\\ \widetilde{g}(r) &:= \frac{1}{\widetilde{w}(r)} \ln\left(1 + \frac{C}{r}\right), \quad \widetilde{g}(0) := 1, \quad r > 0.\end{aligned}$$

Note that $C := \sup_{\mathbf{s}\in\mathbb{S}_+} \|\boldsymbol{\theta}\|^2_{\mathbf{s}\diamond\Sigma}/2$ is a finite constant because $\mathbf{u} \mapsto \|\boldsymbol{\theta}\|^2_{\mathbf{u}\diamond\Sigma}$, $\mathbf{u} \in [0,\infty)^n_*$, is a continuous function on the compact set $\mathbb{S}_+ := \mathbb{S} \cap [0,\infty)^n$. Thus, we get

$$g_{\boldsymbol{\theta}}(\mathbf{u}) = \frac{1}{w(\mathbf{u})} \ln\left(1 + \frac{\|\boldsymbol{\theta}\|^2_{(\mathbf{u}/\|\mathbf{u}\|)\diamond\Sigma}}{2\|\mathbf{u}\|}\right) \le \widetilde{g}(\|\mathbf{u}\|).$$

for all $\mathbf{u} \in [0,\infty)^n_*$. Consider the case $r \ge 1$. Here, we have $\widetilde{w}(r) = r^{-1}$, so that $\widetilde{g}(r) = r\ln(1 + C/r) \le C$. Now consider $0 < r < 1$. Here, we have $\widetilde{w}(r) = 1 - \ln(r)$, so that $\widetilde{g}(r) \to \ln(1+C)$ as $r \to 1$ and $\widetilde{g}(r) \to 1$ as $r \downarrow 0$, which are both finite limits. Thus, $\widetilde{g}$ is uniformly bounded on $[0, \infty)$, and so is $g_{\boldsymbol{\theta}}$ on $[0, \infty)^n_*$.

As $k \to \infty$, recalling that $\mathcal{U}^1_k \xrightarrow{D} \mathcal{U}^1$ and applying the portmanteau lemma (see Lemma 2.2 in [vdV98]) to the continuous and bounded function $g_{\boldsymbol{\theta}}$, we have

$$\Psi_{\mathbf{Y}^{(k)}}(\boldsymbol{\theta}) = -I \int_{[0,\infty)^n_*} g_{\boldsymbol{\theta}}(\mathbf{u})\mathcal{U}^1_k(\mathrm{d}\mathbf{u}) \to -I \int_{[0,\infty)^n_*} g_{\boldsymbol{\theta}}(\mathbf{u})\mathcal{U}^1(\mathrm{d}\mathbf{u}) = \Psi_{\mathbf{Y}}(\boldsymbol{\theta}).$$

Thus, $\mathbf{Y}$ is SD as the class of SD distributions is closed under convergence in distribution by Lemma 5.1.2. □

5.3 Necessary Conditions

This section formulates a converse to Theorem 5.2.2, giving necessary conditions for a VGG^n process to be SD.

Throughout this section, we use the following setup. Assume $n \geq 2$ and $\mathbf{Y} \sim VGG^n(\mathbf{d}, \boldsymbol{\mu}, \Sigma, \mathcal{U})$, where $\Sigma \in \mathbb{R}^{n\times n}$ is an invertible covariance matrix. Thus, $\mathbf{Y} \stackrel{D}{=} \mathbf{B} \odot \mathbf{T}$, where $\mathbf{B} \sim BM^n(\boldsymbol{\mu}, \Sigma)$ and $\mathbf{T} \sim GGC^n_S(\mathbf{d}, \mathcal{T})$. With the notation in Remark 3.2.5, we let $J = \{1, \ldots, n\}$ and $\mathbf{Y}_J \stackrel{D}{=} \mathbf{B} \odot \mathbf{T}_J \sim VGG^n(\mathbf{0}, \boldsymbol{\mu}, \Sigma, \mathcal{U}_J)$ with Lévy measure $\mathcal{Y}_J$ supported on $V_J = (\mathbb{R}_*)^n$. Since Σ is invertible, Theorem 3.2.6 says that $\mathcal{Y}_J$ has Lévy density $\mathrm{d}\mathcal{Y}_J/\mathrm{d}\mathbf{x}$ determined by (3.2.5), where $\mathrm{d}\mathbf{x}$ is the Lebesgue measure on $\mathbb{R}^n$. Recall that $\mathbb{S}^* := \mathbb{S} \cap (\mathbb{R}_*)^n$. Define

$$\mathfrak{H}_{\mathbf{s}}(r) := r^n \frac{\mathrm{d}\mathcal{Y}_J}{\mathrm{d}\mathbf{x}}(r\mathbf{s}), \quad r > 0, \quad \mathbf{s} \in \mathbb{S}^*. \tag{5.3.1}$$

Now we can state the criterion that we will use to prove non-self-decomposability. This is based on Proposition 1 and analogous to Proposition 3 (ii) in [Gri07b], though applied in the context of VGG^n processes. Recall that $\mathrm{d}\mathbf{s}$ denotes the $(n-1)$-dimensional Lebesgue surface measure.

Lemma 5.3.1. *Let $n \geq 2$ and $\mathbf{Y} \sim VGG^n(\mathbf{d}, \boldsymbol{\mu}, \Sigma, \mathcal{U})$, where $|\Sigma| \neq 0$. Then $\mathbf{Y}$ is not SD provided there exists a Borel set $\mathbb{B} \subseteq \mathbb{S}^*$ of strictly positive $(n-1)$-dimensional Lebesgue surface measure such that, for all $\mathbf{s} \in \mathbb{B}$, $r \mapsto \mathfrak{H}_{\mathbf{s}}(r)$ defined in (5.3.1) is strictly increasing at some $r_0 \in (0, \infty)$.*

Proof. By Theorem 3.2.6, then followed by Lemma A.3.3, the Lévy measure of $\mathbf{Y}_J$ is

$$\mathcal{Y}_J(A) = \int_{(\mathbb{R}_*)^n} \mathbf{1}_A(\mathbf{x}) \frac{\mathrm{d}\mathcal{Y}_J}{\mathrm{d}\mathbf{x}}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\mathbb{S}^*}\int_{(0,\infty)} \mathbf{1}_A(r\mathbf{s})\mathfrak{H}_{\mathbf{s}}(r)\,\frac{\mathrm{d}r}{r}\mathrm{d}\mathbf{s} \tag{5.3.2}$$

for all Borel sets $A \subseteq (\mathbb{R}_*)^n$.

For the purpose of contradiction, assume $\mathbf{Y}$ is SD. Thus, by Lemma 5.1.3, there exists a finite measure $\mathcal{S}$ on $\mathbb{S}$ and a nonnegative function $k(\mathbf{s}, r)$ such that

$$\mathcal{Y}(\widetilde{A}) = \int_{\mathbb{S}} \int_{(0,\infty)} \mathbf{1}_{\widetilde{A}}(r\mathbf{s}) k(\mathbf{s}, r) \frac{\mathrm{d}r}{r} \mathcal{S}(\mathrm{d}\mathbf{s}), \tag{5.3.3}$$

for all Borel sets $\widetilde{A} \subseteq \mathbb{R}^n_*$, with $r \mapsto k(\mathbf{s}, r)$ being nonincreasing for all $\mathbf{s} \in \mathbb{S}$. Due to Theorem 3.2.6, $\mathcal{Y}_J(A) = \mathcal{Y}(A)$ for all Borel sets $A \subseteq (\mathbb{R}_*)^n$. So both (5.3.2) and (5.3.3) give polar decompositions of the restriction of $\mathcal{Y}$ to $(\mathbb{R}_*)^n$. However, the polar decomposition is unique up to scaling by a constant with respect to r (see Remark 15.12 (ii) in [Sat99]), so there exists a measurable function $0 < c(\mathbf{s}) < \infty$, $\mathbf{s} \in \mathbb{S}^*$, such that

$$c(\mathbf{s}) k(\mathbf{s}, r) = \mathfrak{H}_{\mathbf{s}}(r), \quad \frac{\mathcal{S}(\mathrm{d}\mathbf{s})}{c(\mathbf{s})} = \mathrm{d}\mathbf{s}, \quad r > 0, \quad \mathbf{s} \in \mathbb{S}^*.$$

Now assume there exists a Borel set $\mathbb{B} \subseteq \mathbb{S}^*$ such that $(\mathrm{d}\mathbf{s})(\mathbb{B}) > 0$, and for all $\mathbf{s} \in \mathbb{B}$, $r \mapsto \mathfrak{H}_{\mathbf{s}}(r)$ is strictly increasing at some $r_0 \in (0, \infty)$. This implies that $\mathcal{S}(\mathbb{B}) > 0$, and for all $\mathbf{s} \in \mathbb{B}$, $r \mapsto k(\mathbf{s}, r)$ is strictly increasing at $r_0 \in (0, \infty)$, which is a contradiction. Thus, $\mathbf{Y}$ cannot be SD. □

We will provide a more explicit formula for (5.3.1). Recall that $\mathbf{Y}_J \sim VGG^n(\mathbf{0}, \boldsymbol{\mu}, \Sigma, \mathcal{U}_J)$, $\boldsymbol{\mu} \in \mathbb{R}^n$, $\Sigma \in \mathbb{R}^{n\times n}$ is an invertible covariance matrix and $\mathcal{U}_J$ is the Thorin measure of $\mathbf{Y}$ restricted to $(0, \infty)^n$. For $\mathbf{u} \in (0, \infty)^n$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^n_*$, introduce

$$\mathfrak{A}_{\boldsymbol{\mu},\Sigma}(\mathbf{x}, \mathbf{u}) := (2\|\mathbf{u}\|^2 + \|\mathbf{u} \diamond \boldsymbol{\mu}\|^2_{(\mathbf{u}\diamond\Sigma)^{-1}})^{1/2} \|\mathbf{x}\|_{(\mathbf{u}\diamond\Sigma)^{-1}}, \tag{5.3.4}$$

$$\mathfrak{D}_{\Sigma}(\mathbf{x}, \mathbf{u}) := \|\mathbf{x}\|^n_{(\mathbf{u}\diamond\Sigma)^{-1}} |\mathbf{u} \diamond \Sigma|^{1/2}, \tag{5.3.5}$$

$$\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{x}, \mathbf{u}) := \langle \mathbf{x}, \mathbf{u} \diamond \boldsymbol{\mu} \rangle_{(\mathbf{u}\diamond\Sigma)^{-1}}, \tag{5.3.6}$$

$$\mathcal{U}_{\mathfrak{D},\mathbf{y}}(\mathrm{d}\mathbf{u}) := \frac{\mathcal{U}_J(\mathrm{d}\mathbf{u})}{\mathfrak{D}_{\Sigma}(\mathbf{y}, \mathbf{u})}. \tag{5.3.7}$$

As explained in Remark 3.2.5, $(\mathbf{u} \diamond \Sigma)^{-1}$ exists, and also $\mathfrak{D}_{\Sigma}(\mathbf{y}, \mathbf{u}) \neq 0$. The variable names $\mathfrak{A}$, $\mathfrak{D}$ and $\mathfrak{E}$ stand for "argument", "denominator" and "exponent", respectively.

Remark 5.3.2. Recall that $\mathbb{S}_{++} := \mathbb{S} \cap (0, \infty)^n$. For $\mathbf{u} \in (0, \infty)^n$, $\mathbf{u}^0 := \mathbf{u}/\|\mathbf{u}\| \in \mathbb{S}_{++}$. With the definitions in (5.3.5) and (5.3.6), we have

$$\mathfrak{D}_{\Sigma}(\mathbf{x}, \mathbf{u}) = \mathfrak{D}_{\Sigma}(\mathbf{x}, \mathbf{u}^0), \quad \mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{x}, \mathbf{u}) = \mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{x}, \mathbf{u}^0). \tag{5.3.8}$$

We also have

$$\|\mathbf{u}\diamond\boldsymbol{\mu}\|^2_{(\mathbf{u}\diamond\Sigma)^{-1}} = \|\mathbf{u}\|\|\mathbf{u}^0\diamond\boldsymbol{\mu}\|^2_{(\mathbf{u}^0\diamond\Sigma)^{-1}}, \tag{5.3.9}$$

$$\|\mathbf{x}\|^2_{(\mathbf{u}\diamond\Sigma)^{-1}} = \frac{\|\mathbf{x}\|^2_{(\mathbf{u}^0\diamond\Sigma)^{-1}}}{\|\mathbf{u}\|}. \tag{5.3.10}$$

These properties will be used in the proofs below.

Let $\mathbf{v}\in(\mathbb{R}_*)^n$. Now the Lévy density of $\mathcal{Y}_J$ given in (3.2.5) and (3.2.6) becomes

$$\frac{\mathrm{d}\mathcal{Y}_J}{\mathrm{d}\mathbf{x}}(\mathbf{v}) = \int_{(0,\infty)^n} \nu_J(\mathbf{v},\mathbf{u})\mathcal{U}(\mathrm{d}\mathbf{u}), \tag{5.3.11}$$

where

$$\nu_J(\mathbf{v},\mathbf{u}) = c_n\frac{\exp(\mathfrak{C}_{\boldsymbol{\mu},\Sigma}(\mathbf{v},\mathbf{u}))}{\mathfrak{D}_\Sigma(\mathbf{v},\mathbf{u})}\mathfrak{K}_{n/2}(\mathfrak{A}_{\boldsymbol{\mu},\Sigma}(\mathbf{v},\mathbf{u})), \tag{5.3.12}$$

$c_n := c_J = 2/(2\pi)^{n/2}$, $\mathbf{u}\in(0,\infty)^n$, $\mathbf{v}\in(\mathbb{R}_*)^n$ and $\mathfrak{K}_{n/2}$ is defined in (1.3.2). Writing $\mathbf{v}=r\mathbf{s}$, $\mathbf{s}\in\mathbb{S}^*$, $r>0$, we introduce

$$\begin{aligned}\mathfrak{h}(\mathbf{u},\mathbf{s},r) &:= r^n\nu_J(r\mathbf{s},\mathbf{u})\\ &= c_n\frac{\exp(r\mathfrak{C}_{\boldsymbol{\mu},\Sigma}(\mathbf{s},\mathbf{u}))}{\mathfrak{D}_\Sigma(\mathbf{s},\mathbf{u})}\mathfrak{K}_{n/2}(r\mathfrak{A}_{\boldsymbol{\mu},\Sigma}(\mathbf{s},\mathbf{u})).\end{aligned} \tag{5.3.13}$$

Then $\mathfrak{H}_\mathbf{s}(r)$ from (5.3.1) becomes

$$\begin{aligned}\mathfrak{H}_\mathbf{s}(r) &= \int_{(0,\infty)^n}\mathfrak{h}(\mathbf{u},\mathbf{s},r)\mathcal{U}(\mathrm{d}\mathbf{u})\\ &= c_n\int_{(0,\infty)^n}\exp(r\mathfrak{C}_{\boldsymbol{\mu},\Sigma}(\mathbf{s},\mathbf{u}))\mathfrak{K}_{n/2}(r\mathfrak{A}_{\boldsymbol{\mu},\Sigma}(\mathbf{s},\mathbf{u}))\mathcal{U}_{\mathfrak{D},\mathbf{s}}(\mathrm{d}\mathbf{u})\end{aligned} \tag{5.3.14}$$

for $\mathbf{s}\in\mathbb{S}^*$, $r>0$. To summarise, $(\mathbf{s},r)\mapsto\mathfrak{H}_\mathbf{s}(r)$, $\mathbf{s}\in\mathbb{S}^*$, $r>0$, is the Lévy density of $\mathcal{Y}_J$ in polar coordinates, and is determined by (5.3.14).

We now state the necessary conditions for self-decomposability. Though the proof is given here, it relies on technical results proven in Section 5.4. The proof extends and refines the arguments of Grigelionis [Gri07b], who showed that the function $r\mapsto\mathfrak{H}_\mathbf{s}(r)$ is strictly increasing for $VGG^{n,1}$ processes, and for sufficiently many $\mathbf{s}\in\mathbb{S}$, by computing its derivative. In light of Lemma 5.3.1, under some assumptions, we show directly without taking derivatives that the function is strictly increasing at the origin for VGG^n processes.

Recall that $x^- := -(x\wedge 0)$, $x\in\mathbb{R}$, and $\prod\mathbf{u} := \prod_{k=1}^n u_k$, $\mathbf{u}=(u_1,\ldots,u_n)\in(0,\infty)^n$. Recall that $\mathfrak{A}_{\boldsymbol{\mu},\Sigma}$, $\mathfrak{D}_\Sigma$, $\mathfrak{C}_{\boldsymbol{\mu},\Sigma}$ and $\mathcal{U}_{\mathfrak{D},\mathbf{s}}$ are defined in (5.3.4)–(5.3.7).

Theorem 5.3.3. *Let* $\mathbf{Y} \sim VGG^n(\mathbf{d}, \boldsymbol{\mu}, \Sigma, \mathcal{U})$. *If* $n \geq 2$ *and* $|\Sigma| \neq 0$, *then* $\mathbf{Y}$ *is not SD provided that one of the following holds:*

(i) there exists a Borel set $\mathbb{B} \subseteq \mathbb{S}^*$ *of strictly positive* $(n-1)$*-dimensional Lebesgue surface measure such that, for all* $\mathbf{s} \in \mathbb{B}$,

$$\int_{(0,\infty)^n} \mathfrak{A}_{\boldsymbol{\mu},\Sigma}(\mathbf{s},\mathbf{u})\mathcal{U}_{\mathfrak{D},\mathbf{s}}(\mathrm{d}\mathbf{u}) < \infty, \tag{5.3.15}$$

$$\int_{(0,\infty)^n} \mathfrak{E}^-_{\boldsymbol{\mu},\Sigma}(\mathbf{s},\mathbf{u})\mathcal{U}_{\mathfrak{D},\mathbf{s}}(\mathrm{d}\mathbf{u}) < \infty, \tag{5.3.16}$$

$$\int_{(0,\infty)^n} \mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{s},\mathbf{u})\mathcal{U}_{\mathfrak{D},\mathbf{s}}(\mathrm{d}\mathbf{u}) \in (0,\infty]; \tag{5.3.17}$$

(ii) $\boldsymbol{\mu} \neq \mathbf{0}$, $\mathcal{U}((0,\infty)^n) > 0$ *and*

$$\int_{(0,\infty)^n} (1 + \|\mathbf{u}\|^{1/2}) \frac{\|\mathbf{u}\|^{n/2}}{(\prod \mathbf{u})^{1/2}} \mathcal{U}(\mathrm{d}\mathbf{u}) < \infty; \tag{5.3.18}$$

(iii) $\boldsymbol{\mu} \neq \mathbf{0}$, *and there exist some* $\boldsymbol{\alpha}_k \in (0,\infty)^n$ *and univariate Thorin measures* $\mathcal{U}_k$, $1 \leq k \leq m$, $m \geq 1$, *satisfying*

$$0 < \int_{(0,\infty)} (1 + u^{1/2})\mathcal{U}_k(\mathrm{d}u) < \infty \tag{5.3.19}$$

such that

$$\mathcal{U}((0,\infty)^n \cap A) = \sum_{k=1}^m \int_{(0,\infty)} \boldsymbol{\delta}_{u\boldsymbol{\alpha}_k}(A)\mathcal{U}_k(\mathrm{d}u)$$

for all Borel sets $A \subseteq [0,\infty)^n_*$.

Proof. (i). Recall the definition of $\mathfrak{H}_\mathbf{s}$ from (5.3.14). We will prove that (5.3.15)–(5.3.17) being satisfied for $\mathbf{s} \in \mathbb{B}$ implies

$$\liminf_{r \downarrow 0} \frac{\mathfrak{H}_\mathbf{s}(2r) - \mathfrak{H}_\mathbf{s}(r)}{r} \in (0,\infty], \tag{5.3.20}$$

which shows that $r \mapsto \mathfrak{H}_\mathbf{s}(r)$ is increasing at 0. Then all the conditions of Lemma 5.3.1 would be satisfied, completing the proof of Part (i).

Recall that $\mathbb{S}_{++} := \mathbb{S} \cap (0,\infty)^n$. For $\mathbf{u} \in (0,\infty)^n$, $\mathbf{u}^0 := \mathbf{u}/\|\mathbf{u}\| \in \mathbb{S}_{++}$. Let $\mathbf{s} \in \mathbb{B}$ and $r > 0$. Let $\mathfrak{E}(\mathbf{u}) := \mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{s},\mathbf{u})$, $\mathfrak{D}(\mathbf{u}) := \mathfrak{D}_\Sigma(\mathbf{s},\mathbf{u})$ and

$$\mathfrak{A}^2(\mathbf{u}) := \mathfrak{A}^2_{\boldsymbol{\mu},\Sigma}(\mathbf{s},\mathbf{u}) = (2\|\mathbf{u}\| + \|\mathbf{u}^0 \diamond \boldsymbol{\mu}\|^2_{(\mathbf{u}^0\diamond\Sigma)^{-1}})\|\mathbf{s}\|^2_{(\mathbf{u}^0\diamond\Sigma)^{-1}}$$

by (5.3.9) and (5.3.10). Let $\mathfrak{h}(\mathbf{u},r) := \mathfrak{h}(\mathbf{u},\mathbf{s},r)$ in (5.3.13) and

$$\mathfrak{h}^*(\mathbf{u},r) := c_n \frac{e^{r\mathfrak{E}(\mathbf{u})}}{\mathfrak{D}(\mathbf{u})}\mathfrak{K}_{n/2}(2r\mathfrak{A}(\mathbf{u})), \quad \mathbf{u} \in (0,\infty)^n, \quad r > 0.$$

Now Lemma 5.4.4 states that

$$\zeta := \inf_{\mathbf{u}^0 \in \mathbb{S}_{++}} \|\mathbf{s}\|^2_{(\mathbf{u}^0 \diamond \Sigma)^{-1}} > 0, \quad \underline{\mathfrak{D}} := \inf_{\mathbf{u}\in(0,\infty)^n} \mathfrak{D}(\mathbf{u}) > 0, \quad \overline{\mathfrak{E}} := \sup_{\mathbf{u}\in(0,\infty)^n} |\mathfrak{E}(\mathbf{u})| < \infty.$$

Note that $\mathcal{U}_{\mathfrak{D}} := \mathcal{U}_{\mathfrak{D},\mathbf{s}}$ satisfies (3.1.1) due to $\underline{\mathfrak{D}} > 0$, so it is a Thorin measure. Since $r \mapsto \mathfrak{K}_{n/2}(r)$ is nonnegative and nonincreasing (see Lemma A.1.1 (i)), and $\mathfrak{A}^2(\mathbf{u}) \geq \zeta\|\mathbf{u}\|$, we have

$$0 \leq \mathfrak{h}^*(\mathbf{u},r) \leq \mathfrak{h}(\mathbf{u},r) \leq c_n \frac{e^{r\overline{\mathfrak{E}}}}{\underline{\mathfrak{D}}}\mathfrak{K}_{n/2}(r\zeta^{1/2}\|\mathbf{u}\|^{1/2}), \quad \mathbf{u} \in (0,\infty)^n,$$

which implies $\mathbf{u} \mapsto \mathfrak{h}(\mathbf{u},r)$ and $\mathbf{u} \mapsto \mathfrak{h}^*(\mathbf{u},r)$ are $\mathcal{U}$-integrable on $(0,\infty)^n$ by (A.1.3). So the integrals

$$\mathfrak{H}(r) := \mathfrak{H}_{\mathbf{s}}(r), \quad \mathfrak{H}^*(r) := \int_{(0,\infty)^n} \mathfrak{h}^*(\mathbf{u},r)\,\mathcal{U}(\mathrm{d}\mathbf{u}), \quad r > 0,$$

are finite.

By substituting in the relevant definitions, we have

$$\frac{\mathfrak{H}(2r) - \mathfrak{H}^*(r)}{r} = c_n \int_{(0,\infty)^n} \frac{1}{r}(e^{r\mathfrak{E}(\mathbf{u})} - 1)e^{r\mathfrak{E}(\mathbf{u})}\mathfrak{K}_{n/2}(2r\mathfrak{A}(\mathbf{u}))\,\mathcal{U}_{\mathfrak{D}}(\mathrm{d}\mathbf{u}). \qquad (5.3.21)$$

We analyse this integral separately for the cases $\mathfrak{E}(\mathbf{u}) \geq 0$ and $\mathfrak{E}(\mathbf{u}) < 0$. Note that $r \mapsto \mathfrak{K}_{n/2}(r)$ has a finite left-hand limit $\mathfrak{K}_{n/2}(0+) > 0$ (see Lemma A.1.1 (iii)). Now using $e^x - 1 \geq x$, $x \in \mathbb{R}$, and Fatou's lemma gives

$$\begin{aligned} &\liminf_{r\downarrow 0} c_n \int_{(0,\infty)^n \cap \{\mathfrak{E}(\mathbf{u})\geq 0\}} \frac{1}{r}(e^{r\mathfrak{E}(\mathbf{u})} - 1)e^{r\mathfrak{E}(\mathbf{u})}\mathfrak{K}_{n/2}(2r\mathfrak{A}(\mathbf{u}))\,\mathcal{U}_{\mathfrak{D}}(\mathrm{d}\mathbf{u}) \\ &\geq c_n\mathfrak{K}_{n/2}(0+) \int_{(0,\infty)^n \cap \{\mathfrak{E}(\mathbf{u})\geq 0\}} \mathfrak{E}(\mathbf{u})\,\mathcal{U}_{\mathfrak{D}}(\mathrm{d}\mathbf{u}), \end{aligned} \qquad (5.3.22)$$

where the RHS is possibly infinite. If $\mathbf{u} \in \{\mathfrak{E}(\mathbf{u}) < 0\}$, then again using $e^x - 1 \geq x$, $x \in \mathbb{R}$, we have

$$\frac{1}{r}|e^{r\mathfrak{E}(\mathbf{u})} - 1|e^{r\mathfrak{E}(\mathbf{u})}\mathfrak{K}_{n/2}(2r\mathfrak{A}(\mathbf{u})) \leq |\mathfrak{E}(\mathbf{u})|\mathfrak{K}_{n/2}(0+) = \mathfrak{E}^-(\mathbf{u})\mathfrak{K}_{n/2}(0+).$$

By (5.3.16), the LHS is $\mathcal{U}_{\mathfrak{D}}$-integrable, so the dominated convergence theorem applies, giving

$$\lim_{r\downarrow 0} c_n \int_{(0,\infty)^n\cap\{\mathfrak{E}(\mathbf{u})<0\}} \frac{1}{r}(e^{r\mathfrak{E}(\mathbf{u})}-1)\, e^{r\mathfrak{E}(\mathbf{u})}\mathfrak{K}_{n/2}(2r\mathfrak{A}(\mathbf{u}))\,\mathcal{U}_{\mathfrak{D}}(\mathrm{d}\mathbf{u})$$
$$= c_n\mathfrak{K}_{n/2}(0+)\int_{(0,\infty)^n\cap\{\mathfrak{E}(\mathbf{u})<0\}} \mathfrak{E}(\mathbf{u})\,\mathcal{U}_{\mathfrak{D}}(\mathrm{d}\mathbf{u}). \tag{5.3.23}$$

To summarise, (5.3.21)–(5.3.23) gives

$$\liminf_{r\downarrow 0}\frac{\mathfrak{H}(2r)-\mathfrak{H}^*(r)}{r} \geq c_n\mathfrak{K}_{n/2}(0+)\int_{(0,\infty)^n}\mathfrak{E}(\mathbf{u})\,\mathcal{U}_{\mathfrak{D}}(\mathrm{d}\mathbf{u}) \in (0,\infty], \tag{5.3.24}$$

where the integral is strictly positive or infinite by (5.3.17).

Now we deal with the remaining term. By substituting in the relevant definitions, we have

$$\frac{\mathfrak{H}(r)-\mathfrak{H}^*(r)}{r} = c_n\int_{(0,\infty)^n}\frac{1}{r}e^{r\mathfrak{E}(\mathbf{u})}(\mathfrak{K}_{n/2}(r\mathfrak{A}(\mathbf{u}))-\mathfrak{K}_{n/2}(2r\mathfrak{A}(\mathbf{u})))\,\mathcal{U}_{\mathfrak{D}}(\mathrm{d}\mathbf{u}).$$

Using (A.1.4), we have

$$0 \leq \frac{\mathfrak{H}(r)-\mathfrak{H}^*(r)}{r} \leq c_n e^{r\overline{\mathfrak{E}}}\int_{(0,\infty)^n}\frac{1}{r}(\mathfrak{K}_{n/2}(r\mathfrak{A}(\mathbf{u}))-\mathfrak{K}_{n/2}(2r\mathfrak{A}(\mathbf{u})))\,\mathcal{U}_{\mathfrak{D}}(\mathrm{d}\mathbf{u})$$
$$\leq \frac{3}{2}c_n e^{r\overline{\mathfrak{E}}}\int_{(0,\infty)^n}(r\mathfrak{A}(\mathbf{u})\mathfrak{K}_\rho(r\mathfrak{A}(\mathbf{u})))\mathfrak{A}(\mathbf{u})\,\mathcal{U}_{\mathfrak{D}}(\mathrm{d}\mathbf{u}), \tag{5.3.25}$$

where $\rho := (n-2)/2 \geq 0$ as $n \geq 2$. By Lemma A.1.1 (xi), $D := \sup_{r>0} r\mathfrak{K}_\rho(r) < \infty$, so the integrand in (5.3.25) is dominated by $D\mathfrak{A}(\mathbf{u})$, which is $\mathcal{U}_{\mathfrak{D}}$-integrable by the assumption (5.3.15). Thus, the dominated convergence theorem is applicable, and noting that $r\mathfrak{A}(\mathbf{u})\mathfrak{K}_\rho(r\mathfrak{A}(\mathbf{u})) \to 0$ as $r\downarrow 0$ for all $\mathbf{u}\in(0,\infty)^n$ by Lemma A.1.1 (viii), it gives

$$\lim_{r\downarrow 0}\frac{\mathfrak{H}(r)-\mathfrak{H}^*(r)}{r} = 0. \tag{5.3.26}$$

Finally,

$$\liminf_{r\downarrow 0}\frac{\mathfrak{H}(2r)-\mathfrak{H}(r)}{r} \geq \liminf_{r\downarrow 0}\frac{\mathfrak{H}(2r)-\mathfrak{H}^*(r)}{r} + \liminf_{r\downarrow 0} - \frac{\mathfrak{H}(r)-\mathfrak{H}^*(r)}{r}$$

so that combining (5.3.24) and (5.3.26) gives (5.3.20), which completes the proof of Part (i).

(ii). Here, we find a Borel set $\mathbb{B}\subseteq\mathbb{S}^*$ satisfying the requirements outlined in Part (i). First, we show that for $\mathbf{s}\in\mathbb{S}^*$, (5.3.18) implies (5.3.15).

For $\mathbf{s} \in \mathbb{S}^*$, Lemma 5.4.4 states that

$$\zeta := \inf_{\mathbf{u}\in\mathbb{S}_{++}} \|\mathbf{s}\|^2_{(\mathbf{u}\diamond\Sigma)^{-1}} > 0, \quad \xi := \sup_{\mathbf{u}\in\mathbb{S}_{++}} \|\mathbf{u}\diamond\boldsymbol{\mu}\|^2_{(\mathbf{u}\diamond\Sigma)^{-1}} < \infty.$$

Let $\mathbf{u} \in (0,\infty)^n$ and $\mathbf{u}^0 := \mathbf{u}/\|\mathbf{u}\|$. By noting that $\|\mathbf{u}\|^n|\mathbf{u}^0\diamond\Sigma| = |\mathbf{u}\diamond\Sigma|$ and using (5.3.9) and (5.3.10), we have

$$\frac{\mathfrak{A}^2_{\boldsymbol{\mu},\Sigma}(\mathbf{s},\mathbf{u})}{\mathfrak{D}^2_{\Sigma}(\mathbf{s},\mathbf{u})} = \frac{2\|\mathbf{u}\| + \|\mathbf{u}^0\diamond\boldsymbol{\mu}\|^2_{(\mathbf{u}^0\diamond\Sigma)^{-1}}}{\|\mathbf{s}\|^{2(n-1)}_{(\mathbf{u}^0\diamond\Sigma)^{-1}}|\mathbf{u}^0\diamond\Sigma|} \leq \frac{2+\xi}{|\Sigma|\zeta^{n-1}}\frac{(1+\|\mathbf{u}\|)\|\mathbf{u}\|^n}{\prod\mathbf{u}},$$

where the last inequality follows by using (5.4.1). Thus, (5.3.15) holds for $\mathbf{s} \in \mathbb{S}^*$ due to the assumption of (5.3.18).

Now we find a Borel set $\mathbb{B} \subseteq \mathbb{S}^*$ with strictly positive Lebesgue surface measure, where both (5.3.16) and (5.3.17) hold. Let $\boldsymbol{\mu}^0 := \boldsymbol{\mu}/\|\boldsymbol{\mu}\| \in \mathbb{R}^n_*$. As we assumed $\boldsymbol{\mu} \neq \mathbf{0}$, Proposition 5.4.11 implies that there exists an open neighbourhood $\mathbb{U}$ of $\mathbf{0}$ such that

$$\inf_{\mathbf{u}\in(0,\infty)^n} \mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{y},\mathbf{u}) > 0, \quad \mathbf{y} \in \boldsymbol{\mu}^0 + \mathbb{U}. \tag{5.3.27}$$

Consequently, the integral in (5.3.16) vanishes for all $\mathbf{s} \in \mathbb{S}^* \cap (\boldsymbol{\mu}^0 + \mathbb{U})$.

Let

$$I(\mathbf{y}) := \int_{(0,\infty)^n} \frac{\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{y},\mathbf{u})}{\mathfrak{D}_{\Sigma}(\mathbf{y},\mathbf{u})}\mathcal{U}(\mathrm{d}\mathbf{u}), \quad \mathbf{y} \in \mathbb{R}^n_*.$$

Using the same arguments as above, we have

$$\frac{\mathfrak{E}^2_{\boldsymbol{\mu},\Sigma}(\mathbf{y},\mathbf{u})}{\mathfrak{D}^2_{\Sigma}(\mathbf{y},\mathbf{u})} \leq \frac{\mathfrak{E}^2_{\boldsymbol{\mu},\Sigma}(\mathbf{y},\mathbf{u})}{\|\mathbf{y}\|^{2n}_{(\mathbf{u}\diamond\Sigma)^{-1}}|\Sigma|}\frac{(1+\|\mathbf{u}\|)\|\mathbf{u}\|^n}{\prod\mathbf{u}}.$$

By Parts (i) and (ii) of Lemma 5.4.6, there exists an open neighbourhood $\mathbb{W}$ of $\mathbf{0}$ such that

$$\sup_{\mathbf{y}\in\boldsymbol{\mu}^{\mathbf{0}}+\mathbb{W}}\ \sup_{\mathbf{u}\in\mathbb{S}_{++}} \frac{\mathfrak{E}^2_{\boldsymbol{\mu},\Sigma}(\mathbf{y},\mathbf{u})}{\|\mathbf{y}\|^{2n}_{(\mathbf{u}\diamond\Sigma)^{-1}}} < \infty.$$

Thus, noting (5.3.18), the dominated convergence theorem is applicable and implies that $\mathbf{y} \mapsto I(\mathbf{y})$ is continuous at $\mathbf{y} = \boldsymbol{\mu}^0$ (see Lemma 16.1 in [Bau92]). Now $I(\boldsymbol{\mu}^0) > 0$ due to (5.3.27) and the assumption that $\mathcal{U}((0,\infty)^n) > 0$. By continuity, there exists another open neighbourhood $\mathbb{V} \subseteq \mathbb{W}$ of $\mathbf{0}$ such that $I(\mathbf{s}) > 0$ for all $\mathbf{s} \in \mathbb{S}^* \cap (\boldsymbol{\mu}^0 + \mathbb{V})$.

To summarise, there exists a Borel set $\mathbb{B} \subseteq \mathbb{S}^* \cap (\boldsymbol{\mu}^0 + \mathbb{U}) \cap (\boldsymbol{\mu}^0 + \mathbb{V})$ with strictly positive Lebesgue surface measure where both (5.3.16) and (5.3.17) holds. Applying Part (i) completes the proof.

(iii). For $1 \le k \le m$, set $u_k := \prod \boldsymbol{\alpha}_k$. Note that $\|u\boldsymbol{\alpha}_k\|^n = u^n\|\boldsymbol{\alpha}_k\|^n$ and $\prod(u\boldsymbol{\alpha}_k) = u^n u_k$, $u > 0$, so we have

$$\begin{aligned}
\int_{(0,\infty)^n} (1+\|\mathbf{u}\|^{1/2})\frac{\|\mathbf{u}\|^{n/2}}{(\prod \mathbf{u})^{1/2}}\mathcal{U}(\mathrm{d}\mathbf{u}) &= \sum_{k=1}^{n}\left(\frac{\|\boldsymbol{\alpha}_k\|^n}{u_k}\right)^{1/2}\int_{(0,\infty)}(1+\|u\boldsymbol{\alpha}_k\|^{1/2})\,\mathcal{U}_k(\mathrm{d}u) \\
&\le \sum_{k=1}^{m}\left(\frac{\|\boldsymbol{\alpha}_k\|^n}{u_k}\right)^{1/2}(1+\|\boldsymbol{\alpha}_k\|^{1/2})\int_{(0,\infty)}(1+u^{1/2})\,\mathcal{U}_k(\mathrm{d}u) \\
&< \infty
\end{aligned}$$

by assumption. Thus (5.3.18) holds, so the proof is completed by Part (ii). □

We make some remarks about the conditions and proof of the above theorem.

Remark 5.3.4. If $\boldsymbol{\mu} = \mathbf{0}$, then the conditions of Part (i) cannot be satisfied because the integral in (5.3.17) vanishes. This is consistent with the sufficient conditions in Theorem 5.2.2. In the proof of Part (i), the arguments still work when $\mathbf{s} \in \mathbb{B}$ is replaced by $\mathbf{v} \in (\mathbb{R}_*)^n$. In the proof of Part (ii), we showed that (5.3.18) implies (5.3.15) for all $\mathbf{s} \in \mathbb{S}^*$. In fact, this is true more generally when $\mathbf{s} \in \mathbb{S}^*$ is replaced by $\mathbf{y} \in \mathbb{R}^n_*$.

While (5.3.15)–(5.3.17) in Part (i) shows a delicate dependency between $\boldsymbol{\mu}$, Σ and $\mathbf{s}$, we use the more robust condition of Parts (ii) and (iii) in the applications of Section 5.5 below. In Part (iii), the Thorin measure is supported on a finite union of rays. In particular, this includes all $WVAG$ processes. Other examples include the $V\mathcal{M}\Gamma^n$ processes in Section 2.5 of [BKMS17].

By Proposition 3.3.1 (iii), the condition (5.3.19) requires that the VGG^n process have paths of bounded variation.

Example 5.3.5. Let $\mathcal{U}_1, \mathcal{U}_2$ be Borel measures defined by

$$\mathcal{U}_1(A) = \int_0^1 \boldsymbol{\delta}_{(\cos(\theta),\sin(\theta))}(A)\,\mathrm{d}\theta, \quad \mathcal{U}_2(A) = \int_0^1 \boldsymbol{\delta}_{(\cos(\theta^2),\sin(\theta^2))}(A)\,\mathrm{d}\theta$$

for any Borel set $A \subseteq [0,\infty)^2_*$. Both of these are Thorin measures as the LHS of (3.1.1) equals 1. For $\mathcal{U}_1, \mathcal{U}_2$, the integral in (5.3.18) becomes

$$2\int_0^1 (\cos(\theta)\sin(\theta))^{-1/2}\,\mathrm{d}\theta < \infty, \quad 2\int_0^1 (\cos(\theta^2)\sin(\theta^2))^{-1/2}\,\mathrm{d}\theta = \infty,$$

respectively. Also, $\mathcal{U}_1((0,\infty)^2) = \mathcal{U}_2((0,\infty)^2) = 1$. So on the basis of Theorem 5.3.3

(ii), when $\boldsymbol{\mu} \neq \mathbf{0}$ and $|\Sigma| \neq 0$, $\mathbf{Y}^{(1)} \sim VGG^2(\mathbf{d}, \boldsymbol{\mu}, \Sigma, \mathcal{U}_1)$ is not SD, while no conclusion can be drawn for $\mathbf{Y}^{(2)} \sim VGG^2(\mathbf{d}, \boldsymbol{\mu}, \Sigma, \mathcal{U}_2)$.

5.4 Technical Results for Proving the Necessary Conditions

In this section, we gives some technical results that are used to prove the necessary conditions in Theorem 5.3.3. Broadly, these relate to analysing the various terms of $\mathfrak{H}_{\mathbf{s}}$ defined in (5.3.14), and in particular, the terms $\mathfrak{A}_{\boldsymbol{\mu},\Sigma}$, $\mathfrak{D}_{\Sigma}$, $\mathfrak{C}_{\boldsymbol{\mu},\Sigma}$ defined in (5.3.4)–(5.3.6).

Throughout this section, $\Sigma = (\Sigma_{kl}) \in \mathbb{R}^{n\times n}$ is always assumed to be an invertible covariance matrix.

5.4.1 Infimums and Supremums of Terms in $\mathfrak{H}$

Recall that $\prod \mathbf{u} := \prod_{k=1}^{n} u_k$ for $\mathbf{u} = (u_1, \dots, u_n) \in (0,\infty)^n$. If $A = (A_{kl}) \in \mathbb{R}^{n\times n}$ and $B = (B_{kl}) \in \mathbb{R}^{n\times n}$, recall that the Hadamard product of A and B is $A * B := (A_{kl}B_{kl}) \in \mathbb{R}^{n\times n}$.

It was shown in Remark 3.2.5 that $\mathbf{u} \diamond \Sigma$, $\mathbf{u} \in (0,\infty)^n$, is invertible. The next lemma is a stronger version of this claim.

Lemma 5.4.1. *If $\mathbf{u} \in (0,\infty)^n$, then*

$$0 < |\Sigma| \leq \inf_{\mathbf{u}\in(0,\infty)^n} \frac{|\mathbf{u} \diamond \Sigma|}{\prod \mathbf{u}} \leq \sup_{\mathbf{u}\in(0,\infty)^n} \frac{|\mathbf{u} \diamond \Sigma|}{\prod \mathbf{u}} \leq \prod_{k=1}^{n} \Sigma_{kk} < \infty. \tag{5.4.1}$$

Proof. The first inequality is due to the invertibility of Σ. Noting that $\mathbf{u} \diamond \Sigma = U * \Sigma$, where $U := (u_k \wedge u_l) \in \mathbb{R}^{n\times n}$, the second and fourth inequalities follow from Oppenheim's and Hadamard's inequalities found in (A.2.1), respectively. □

We give a coordinate permutation formula.

Lemma 5.4.2. *If $P \in \mathbb{R}^{n\times n}$ is a permutation matrix, then*

$$\langle \mathbf{x}, \mathbf{y} \rangle_{(\mathbf{u}\diamond\Sigma)^{-1}} = \langle \mathbf{x}P, \mathbf{y}P \rangle_{((\mathbf{u}P)\diamond(P'\Sigma P))^{-1}} \tag{5.4.2}$$

for all $\mathbf{u} \in (0,\infty)^n$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Proof. Let $\langle (1), \dots, (n) \rangle$ be the permutation associated with P, so the (k,l) element of $P'\Sigma P$ is $\Sigma_{(k)(l)}$. Then

$$(\mathbf{u}P) \diamond (P'\Sigma P) = P'(\mathbf{u} \diamond \Sigma)P \tag{5.4.3}$$

because the (k,l) elements of both sides are $(u_{(k)} \wedge u_{(l)})\Sigma_{(k)(l)}$. Thus,

$$\begin{aligned}\langle \mathbf{x}P, \mathbf{y}P\rangle_{((\mathbf{u}P)\diamond(P'\Sigma P))^{-1}} &= \mathbf{x}P((\mathbf{u}P) \diamond (P'\Sigma P))^{-1}P'\mathbf{y}' \\ &= \mathbf{x}P(P'(\mathbf{u} \diamond \Sigma)P)^{-1}P'\mathbf{y}' \\ &= \mathbf{x}(\mathbf{u} \diamond \Sigma)\mathbf{y}',\end{aligned}$$

where the last line matches the LHS of (5.4.2). $\square$

Let $(0,\infty)^n_{\le} := \{\mathbf{u} = (u_1, \ldots, u_n) \in (0,\infty)^n : u_1 \le \cdots \le u_n\}$. Recall that $\mathfrak{E}_{\boldsymbol{\mu},\Sigma}$ is defined in (5.3.6). For $n \ge 2$, we introduce notation to express $\mathbf{u} \in (0,\infty)^n_{\le}$ and an invertible covariance matrix $\Sigma \in \mathbb{R}^{n\times n}$ with terms of one lower dimension by setting

$$\begin{aligned}\mathbf{u} = (\widetilde{\mathbf{u}}, u), \quad \widetilde{\mathbf{u}} \in (0,\infty)^{n-1}_{\le}, \quad u \in (0,\infty), \\ \Sigma = \begin{pmatrix} \widetilde{\Sigma} & \widetilde{\boldsymbol{\sigma}}' \\ \widetilde{\boldsymbol{\sigma}} & \sigma \end{pmatrix}, \quad \widetilde{\Sigma} \in \mathbb{R}^{(n-1)\times(n-1)}, \quad \widetilde{\boldsymbol{\sigma}} \in \mathbb{R}^{n-1}, \quad \sigma > 0.\end{aligned}$$

Also, let $\mathbf{x} = (\widetilde{\mathbf{x}}, x)$, $\mathbf{y} = (\widetilde{\mathbf{y}}, y)$, $\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}} \in \mathbb{R}^{n-1}$, $x, y \in \mathbb{R}$. We call this *dimension reduction notation*. Note that $\widetilde{\Sigma} \in \mathbb{R}^{(n-1)\times(n-1)}$ must be invertible due to Sylvester's criterion. Next we give a dimension reduction formula.

Lemma 5.4.3. *For $n \ge 2$, $\mathbf{u} \in (0,\infty)^n_{\le}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have*

$$\langle \mathbf{x}, \mathbf{y}\rangle_{(\mathbf{u}\diamond\Sigma)^{-1}} = \langle \widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}\rangle_{(\widetilde{\mathbf{u}}\diamond\widetilde{\Sigma})^{-1}} + \frac{|\widetilde{\mathbf{u}} \diamond \widetilde{\Sigma}|}{|\mathbf{u} \diamond \Sigma|}(x - \mathfrak{E}_{\widetilde{\boldsymbol{\sigma}},\widetilde{\Sigma}}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{u}}))(y - \mathfrak{E}_{\widetilde{\boldsymbol{\sigma}},\widetilde{\Sigma}}(\widetilde{\mathbf{y}}, \widetilde{\mathbf{u}})). \tag{5.4.4}$$

Proof. Since $\mathbf{u} \in (0,\infty)^n_{\le}$, we can write

$$\mathbf{u} \diamond \Sigma = \begin{pmatrix} \widetilde{\mathbf{u}} \diamond \widetilde{\Sigma} & (\widetilde{\mathbf{u}} \diamond \widetilde{\boldsymbol{\sigma}})' \\ \widetilde{\mathbf{u}} \diamond \widetilde{\boldsymbol{\sigma}} & u\sigma \end{pmatrix}, \quad (\mathbf{u} \diamond \Sigma)^{-1} = \begin{pmatrix} \widetilde{A} & \widetilde{\boldsymbol{\alpha}}' \\ \widetilde{\boldsymbol{\alpha}} & a \end{pmatrix} \tag{5.4.5}$$

for some $\widetilde{A} \in \mathbb{R}^{(n-1)\times(n-1)}$, $\widetilde{\boldsymbol{\alpha}} \in \mathbb{R}^{n-1}$ and $a > 0$. Let $I_n \in \mathbb{R}^{n\times n}$ be the identity matrix. Now using (5.4.5) to expand $(\mathbf{u} \diamond \Sigma)(\mathbf{u} \diamond \Sigma)^{-1} = I_n$ in block matrix form and taking the top-left $(n-1) \times (n-1)$ submatrix and the top-right $(n-1) \times 1$ submatrix of both sides, we get

$$(\widetilde{\mathbf{u}} \diamond \widetilde{\Sigma})\widetilde{A} + (\widetilde{\mathbf{u}} \diamond \widetilde{\boldsymbol{\sigma}})'\widetilde{\boldsymbol{\alpha}} = I_{n-1}, \tag{5.4.6}$$

$$(\widetilde{\mathbf{u}} \diamond \widetilde{\Sigma})\widetilde{\boldsymbol{\alpha}}' + a(\widetilde{\mathbf{u}} \diamond \widetilde{\boldsymbol{\sigma}})' = \mathbf{0}'. \tag{5.4.7}$$

The LHS of (5.4.4) can be written as

$$\langle \mathbf{x}, \mathbf{y} \rangle_{(\mathbf{u}\diamond\Sigma)^{-1}} = (\widetilde{\mathbf{x}}, x) \begin{pmatrix} \widetilde{A} & \widetilde{\boldsymbol{\alpha}}' \\ \widetilde{\boldsymbol{\alpha}} & a \end{pmatrix} \begin{pmatrix} \widetilde{\mathbf{y}}' \\ y \end{pmatrix} = \widetilde{\mathbf{x}} \widetilde{A} \widetilde{\mathbf{y}}' + (y\widetilde{\mathbf{x}} + x\widetilde{\mathbf{y}})\widetilde{\boldsymbol{\alpha}}' + axy. \tag{5.4.8}$$

The first term on the RHS of (5.4.8) becomes

$$\begin{aligned} \widetilde{\mathbf{x}} \widetilde{A} \widetilde{\mathbf{y}}' &= \widetilde{\mathbf{x}}((\widetilde{\mathbf{u}} \diamond \widetilde{\Sigma})^{-1} - (\widetilde{\mathbf{u}} \diamond \widetilde{\Sigma})^{-1}(\widetilde{\mathbf{u}} \diamond \widetilde{\boldsymbol{\sigma}})'\widetilde{\boldsymbol{\alpha}})\widetilde{\mathbf{y}}' \\ &= \widetilde{\mathbf{x}}(\widetilde{\mathbf{u}} \diamond \widetilde{\Sigma})^{-1}\widetilde{\mathbf{y}}' + a\widetilde{\mathbf{x}}(\widetilde{\mathbf{u}} \diamond \widetilde{\Sigma})^{-1}(\widetilde{\mathbf{u}} \diamond \widetilde{\boldsymbol{\sigma}})'(\widetilde{\mathbf{u}} \diamond \widetilde{\boldsymbol{\sigma}})(\widetilde{\mathbf{u}} \diamond \widetilde{\Sigma})^{-1}\widetilde{\mathbf{y}}' \\ &= \langle \widetilde{\mathbf{x}}, \widetilde{\mathbf{y}} \rangle_{(\widetilde{\mathbf{u}}\diamond\widetilde{\Sigma})^{-1}} + a\langle \widetilde{\mathbf{x}}, \widetilde{\mathbf{u}} \diamond \widetilde{\boldsymbol{\sigma}} \rangle_{(\widetilde{\mathbf{u}}\diamond\widetilde{\Sigma})^{-1}} \langle \widetilde{\mathbf{y}}, \widetilde{\mathbf{u}} \diamond \widetilde{\boldsymbol{\sigma}} \rangle_{(\widetilde{\mathbf{u}}\diamond\widetilde{\Sigma})^{-1}}, \end{aligned} \tag{5.4.9}$$

where the first line was obtained by substituting $\widetilde{A}$ from (5.4.6), noting that $\widetilde{\mathbf{u}} \diamond \widetilde{\Sigma}$ is an invertible covariance matrix since $\mathbf{u} \diamond \Sigma$ is, and the second line was obtained by substituting $\widetilde{\boldsymbol{\alpha}}$ from (5.4.7). The second term on the RHS of (5.4.8) is

$$\begin{aligned} (y\widetilde{\mathbf{x}} + x\widetilde{\mathbf{y}})\widetilde{\boldsymbol{\alpha}}' &= -a(y\widetilde{\mathbf{x}} + x\widetilde{\mathbf{y}})(\widetilde{\mathbf{u}} \diamond \widetilde{\Sigma})^{-1}(\widetilde{\mathbf{u}} \diamond \widetilde{\boldsymbol{\sigma}})' \\ &= -ay\langle \widetilde{\mathbf{x}}, \widetilde{\mathbf{u}} \diamond \widetilde{\boldsymbol{\sigma}} \rangle_{(\widetilde{\mathbf{u}}\diamond\widetilde{\Sigma})^{-1}} - ax\langle \widetilde{\mathbf{y}}, \widetilde{\mathbf{u}} \diamond \widetilde{\boldsymbol{\sigma}} \rangle_{(\widetilde{\mathbf{u}}\diamond\widetilde{\Sigma})^{-1}}, \end{aligned} \tag{5.4.10}$$

by substituting $\widetilde{\boldsymbol{\alpha}}'$ from (5.4.7).

Now note that $a = |\widetilde{\mathbf{u}} \diamond \widetilde{\Sigma}|/|\mathbf{u} \diamond \Sigma|$ by Cramer's rule for matrix inverses. Thus, combining (5.4.8)–(5.4.10) gives the RHS of (5.4.4). □

Now we can determine the infimums and supremums of the terms found in the expression for $\mathfrak{H}_{\mathbf{s}}$ in (5.3.14). Recall that Σ is an invertible covariance matrix, and that $\mathfrak{D}_\Sigma$ and $\mathfrak{E}_{\boldsymbol{\mu},\Sigma}$ are defined in (5.3.5) and (5.3.6), respectively. Also, $\mathbb{S}_{++} := \mathbb{S} \cap (0, \infty)^n$. The meaning of the notation $\mathbb{S}$ and $\mathbb{S}_{++}$ is understood in the usual way when used in the context of $\mathbb{R}^m$, $m \neq n$.

Lemma 5.4.4. *Let $\boldsymbol{\mu}, \mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^n_*$ and $\mathbf{v} \in (\mathbb{R}_*)^n$. Then*

$$\inf_{\mathbf{u}\in\mathbb{S}_{++}} \|\mathbf{y}\|_{(\mathbf{u}\diamond\Sigma)^{-1}} > 0, \tag{5.4.11}$$

$$\inf_{\mathbf{u}\in(0,\infty)^n} \mathfrak{D}_\Sigma(\mathbf{v}, \mathbf{u}) = \inf_{\mathbf{u}\in\mathbb{S}_{++}} \mathfrak{D}_\Sigma(\mathbf{v}, \mathbf{u}) > 0, \tag{5.4.12}$$

$$\sup_{\mathbf{u}\in\mathbb{S}_{++}} \|\mathbf{u} \diamond \boldsymbol{\mu}\|_{(\mathbf{u}\diamond\Sigma)^{-1}} < \infty, \tag{5.4.13}$$

$$\sup_{\mathbf{u}\in(0,\infty)^n} |\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{x}, \mathbf{u})| = \sup_{\mathbf{u}\in\mathbb{S}_{++}} |\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{x}, \mathbf{u})| < \infty. \tag{5.4.14}$$

Proof. Taking the supremum and infimum of $\mathfrak{D}_\Sigma$ and $\mathfrak{E}_{\boldsymbol{\mu},\Sigma}$ over $\mathbf{u} \in (0, \infty)^n$ is the same as taking it over $\mathbf{u} \in \mathbb{S}_{++}$ due to (5.3.8).

Moreover, we can assume without loss of generality that $\mathbf{u} \in \mathbb{S} \cap (0, \infty)^n_{\le}$ as all the quantities in the lemma are invariant under permutations as determined by (5.4.2). To see this invariance for $\mathfrak{D}_\Sigma(\mathbf{v}, \mathbf{u})$, we also need to note that $|(\mathbf{u}P) \diamond (P'\Sigma P)| = |\mathbf{u} \diamond \Sigma|$ by (5.4.3).

Using dimension reduction notation, we have $\mathbf{u} = (\widetilde{\mathbf{u}}, u) \in \mathbb{S} \cap (0, \infty)^n_{\le}$ satisfying

$$0 < \|\widetilde{\mathbf{u}}\| \le 1, \quad n^{-1/2} \le u < 1$$

due to $\|\widetilde{\mathbf{u}}\|^2 \le \|\mathbf{u}\|^2 \le nu^2$ and $\mathbf{u} \in \mathbb{S} \cap (0, \infty)^n_{\le}$. Let $\widetilde{\mathbf{u}}^0 := \widetilde{\mathbf{u}}/\|\widetilde{\mathbf{u}}\| \in \mathbb{S} \cap (0, \infty)^{n-1}_{\le}$. For $n \ge 2$, let

$$C(\Sigma) := \frac{|\widetilde{\Sigma}|}{\prod_{k=1}^n \Sigma_{kk}}, \quad D(\Sigma) := \frac{n^{1/2} \prod_{k=1}^{n-1} \Sigma_{kk}}{|\widetilde{\Sigma}|}. \tag{5.4.15}$$

Using (5.4.1) and noting that $n^{-1/2} \le u < 1$, we have

$$0 < C(\Sigma) < \frac{|\widetilde{\Sigma}|}{u \prod_{k=1}^n \Sigma_{kk}} \le \frac{|\widetilde{\mathbf{u}} \diamond \widetilde{\Sigma}|}{|\mathbf{u} \diamond \Sigma|} \le \frac{u \prod_{k=1}^n \Sigma_{kk}}{|\widetilde{\Sigma}|} \le D(\Sigma) < \infty. \tag{5.4.16}$$

With this setup, we can now prove each inequality using mathematical induction. *Proof of* (5.4.11). Let $\zeta_{\mathbf{y}} := \inf_{\mathbf{u} \in \mathbb{S}_{++}} \|\mathbf{y}\|^2_{(\mathbf{u} \diamond \Sigma)^{-1}}$. If $n = 1$, then $u = 1$ and $\zeta_y = y^2/\Sigma > 0$.

Suppose $n \ge 2$. Let $\mathbf{y} = (\widetilde{\mathbf{y}}, y) \in \mathbb{R}^n_*$. Using (5.4.4), the LHS of (5.4.16), (5.3.10) and $\|\widetilde{\mathbf{u}}\| \le 1$, in this order, we have

$$\|\mathbf{y}\|^2_{(\mathbf{u} \diamond \Sigma)^{-1}} \ge \|\widetilde{\mathbf{y}}\|^2_{(\widetilde{\mathbf{u}} \diamond \widetilde{\Sigma})^{-1}} + C(\Sigma)(y - \mathfrak{E}_{\widetilde{\boldsymbol{\sigma}}, \widetilde{\Sigma}}(\widetilde{\mathbf{y}}, \widetilde{\mathbf{u}}))^2 \tag{5.4.17}$$

$$\ge \|\widetilde{\mathbf{y}}\|^2_{(\widetilde{\mathbf{u}} \diamond \widetilde{\Sigma})^{-1}} \tag{5.4.18}$$

$$= \frac{\|\widetilde{\mathbf{y}}\|^2_{(\widetilde{\mathbf{u}}^0 \diamond \widetilde{\Sigma})^{-1}}}{\|\widetilde{\mathbf{u}}\|}$$

$$\ge \|\widetilde{\mathbf{y}}\|^2_{(\widetilde{\mathbf{u}}^0 \diamond \widetilde{\Sigma})^{-1}} \tag{5.4.19}$$

$$\ge \zeta_{\widetilde{\mathbf{y}}}, \tag{5.4.20}$$

where $\zeta_{\widetilde{\mathbf{y}}} := \inf_{\widetilde{\mathbf{u}} \in \mathbb{S}_{++}} \|\widetilde{\mathbf{y}}\|^2_{(\widetilde{\mathbf{u}} \diamond \widetilde{\Sigma})^{-1}} > 0$ by the inductive hypothesis if $\widetilde{\mathbf{y}} \ne \mathbf{0}$. Otherwise, if $\widetilde{\mathbf{y}} = \mathbf{0}$, then $y \ne 0$, so that (5.4.17) becomes $\|\mathbf{y}\|^2_{(\mathbf{u} \diamond \Sigma)^{-1}} \ge C(\Sigma)y^2 > 0$. Thus, $\zeta_{\mathbf{y}} > 0$ in both cases.

Proof of (5.4.12). Let $\underline{\mathfrak{D}}_{\mathbf{v}} := \inf_{\mathbf{u} \in \mathbb{S}_{++}} \mathfrak{D}^2_\Sigma(\mathbf{v}, \mathbf{u})$. If $n = 1$, then $\underline{\mathfrak{D}}_v = v^2 > 0$.

Suppose $n \ge 2$. Let $\mathbf{v} = (\widetilde{\mathbf{v}}, v) \in (\mathbb{R}_*)^n$ using dimension reduction notation. Let $\underline{\mathfrak{D}}_{\widetilde{\mathbf{v}}} := \inf_{\widetilde{\mathbf{u}} \in \mathbb{S}_{++}} \|\widetilde{\mathbf{v}}\|^{2(n-1)}_{(\widetilde{\mathbf{u}} \diamond \widetilde{\Sigma})^{-1}} |\widetilde{\mathbf{u}} \diamond \widetilde{\Sigma}|$. Recalling the definition of $\mathfrak{D}^2_\Sigma(\mathbf{v}, \mathbf{u})$ in (5.3.5), we have

$$
\begin{aligned}
\mathfrak{D}^2_\Sigma(\mathbf{v},\mathbf{u}) &\geq \frac{\|\mathbf{v}\|^{2n}_{(\mathbf{u}\diamond\Sigma)^{-1}}|\mathbf{u}\diamond\Sigma|}{\|\widetilde{\mathbf{v}}\|^{2(n-1)}_{(\widetilde{\mathbf{u}}^0\diamond\widetilde{\Sigma})^{-1}}|\widetilde{\mathbf{u}}^0\diamond\widetilde{\Sigma}|}\mathfrak{D}_{\widetilde{\mathbf{v}}} \\
&= \frac{\|\mathbf{v}\|^{2n}_{(\mathbf{u}\diamond\Sigma)^{-1}}|\mathbf{u}\diamond\Sigma|}{\|\widetilde{\mathbf{v}}\|^{2(n-1)}_{(\widetilde{\mathbf{u}}\diamond\widetilde{\Sigma})^{-1}}|\widetilde{\mathbf{u}}\diamond\widetilde{\Sigma}|}\mathfrak{D}_{\widetilde{\mathbf{v}}} \\
&\geq \|\widetilde{\mathbf{v}}\|^2_{(\widetilde{\mathbf{u}}\diamond\widetilde{\Sigma})^{-1}}(D(\Sigma))^{-1}\mathfrak{D}_{\widetilde{\mathbf{v}}} \\
&\geq \zeta_{\widetilde{\mathbf{v}}}(D(\Sigma))^{-1}\mathfrak{D}_{\widetilde{\mathbf{v}}} \\
&> 0,
\end{aligned}
$$

where the second line follows from (5.3.8), the third line follows from (5.4.18) and the reciprocal of the RHS of (5.4.16) and the fourth line follows from (5.4.20). Thus, by the inductive hypothesis $\mathfrak{D}_{\mathbf{v}} > 0$.

Proof of (5.4.13). Let $\xi_{\boldsymbol{\mu}} := \sup_{\mathbf{u}\in\mathbb{S}_{++}}\|\mathbf{u}\diamond\boldsymbol{\mu}\|^2_{(\mathbf{u}\diamond\Sigma)^{-1}}$. If $n=1$, then $\xi_\mu = |\mu|/\Sigma < \infty$.

Suppose $n \geq 2$. Let $\boldsymbol{\mu} = (\widetilde{\boldsymbol{\mu}}, \mu) \in \mathbb{R}^n$ using dimension reduction notation. Using (5.4.4), the RHS of (5.4.16) and noting that $(a-b)^2 \leq 2(a^2+b^2)$, $a, b \in \mathbb{R}$, we have

$$
\|\mathbf{u}\diamond\boldsymbol{\mu}\|^2_{(\mathbf{u}\diamond\Sigma)^{-1}} \leq \|\widetilde{\mathbf{u}}\diamond\widetilde{\boldsymbol{\mu}}\|^2_{(\widetilde{\mathbf{u}}\diamond\widetilde{\Sigma})^{-1}} + 2D(\Sigma)((u\mu)^2 + \mathfrak{E}^2_{\widetilde{\boldsymbol{\sigma}},\widetilde{\Sigma}}(\widetilde{\mathbf{u}}\diamond\widetilde{\boldsymbol{\mu}},\widetilde{\mathbf{u}})). \tag{5.4.21}
$$

We deal with each term on the RHS. Firstly, by (5.3.9) and noting that $\|\widetilde{\mathbf{u}}\| \leq 1$, we have

$$
\|\widetilde{\mathbf{u}}\diamond\widetilde{\boldsymbol{\mu}}\|^2_{(\widetilde{\mathbf{u}}\diamond\widetilde{\Sigma})^{-1}} = \|\mathbf{u}\|\|\widetilde{\mathbf{u}}^0\diamond\widetilde{\boldsymbol{\mu}}\|^2_{(\widetilde{\mathbf{u}}^0\diamond\widetilde{\Sigma})^{-1}} \leq \|\widetilde{\mathbf{u}}^0\diamond\widetilde{\boldsymbol{\mu}}\|^2_{(\widetilde{\mathbf{u}}^0\diamond\widetilde{\Sigma})^{-1}} \leq \xi_{\widetilde{\boldsymbol{\mu}}},
$$

where $\xi_{\widetilde{\boldsymbol{\mu}}} := \sup_{\widetilde{\mathbf{u}}\in\mathbb{S}_{++}}\|\widetilde{\mathbf{u}}\diamond\widetilde{\boldsymbol{\mu}}\|^2_{(\widetilde{\mathbf{u}}\diamond\widetilde{\Sigma})^{-1}}$. Secondly, $(u\mu)^2 < \mu^2$ as $u < 1$. Thirdly, recalling the definition of $\mathfrak{E}_{\boldsymbol{\mu},\Sigma}$ in (5.3.6) and using the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
|\mathfrak{E}_{\widetilde{\boldsymbol{\sigma}},\widetilde{\Sigma}}(\widetilde{\mathbf{u}}\diamond\widetilde{\boldsymbol{\mu}},\widetilde{\mathbf{u}})|^2 &= |\langle\widetilde{\mathbf{u}}\diamond\widetilde{\boldsymbol{\mu}},\widetilde{\mathbf{u}}\diamond\widetilde{\boldsymbol{\sigma}}\rangle_{(\widetilde{\mathbf{u}}\diamond\widetilde{\Sigma})^{-1}}|^2 \\
&\leq \|\widetilde{\mathbf{u}}\diamond\widetilde{\boldsymbol{\mu}}\|^2_{(\widetilde{\mathbf{u}}\diamond\widetilde{\Sigma})^{-1}}\|\widetilde{\mathbf{u}}\diamond\widetilde{\boldsymbol{\sigma}}\|^2_{(\widetilde{\mathbf{u}}\diamond\widetilde{\Sigma})^{-1}} \\
&\leq \|\widetilde{\mathbf{u}}\|^2\|\widetilde{\mathbf{u}}^0\diamond\widetilde{\boldsymbol{\mu}}\|^2_{(\widetilde{\mathbf{u}}^0\diamond\widetilde{\Sigma})^{-1}}\|\widetilde{\mathbf{u}}^0\diamond\widetilde{\boldsymbol{\sigma}}\|^2_{(\widetilde{\mathbf{u}}^0\diamond\widetilde{\Sigma})^{-1}} \\
&\leq \xi_{\widetilde{\boldsymbol{\mu}}}\xi_{\widetilde{\boldsymbol{\sigma}}}
\end{aligned} \tag{5.4.22}
$$

To summarise,

$$
\|\mathbf{u}\diamond\boldsymbol{\mu}\|^2_{(\mathbf{u}\diamond\Sigma)^{-1}} \leq \xi_{\widetilde{\boldsymbol{\mu}}} + 2D(\Sigma)(\mu^2 + \xi_{\widetilde{\boldsymbol{\mu}}}\xi_{\widetilde{\boldsymbol{\sigma}}}) < \infty
$$

by the inductive hypothesis. Thus, $\xi_{\boldsymbol{\mu}} < \infty$.

Proof of (5.4.14). Let $\overline{\mathfrak{E}}_{\boldsymbol{\mu},\mathbf{x}} := \sup_{\mathbf{u}\in\mathbb{S}_{++}} |\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{x},\mathbf{u})|$. If $n=1$, then $\overline{\mathfrak{E}}_{\mu,x} = |x\mu|/\Sigma < \infty$.

Suppose $n \geq 2$. Let $\overline{\mathfrak{E}}_{\widetilde{\boldsymbol{\mu}},\widetilde{\mathbf{x}}} := \sup_{\widetilde{\mathbf{u}}\in\mathbb{S}_{++}} |\mathfrak{E}_{\widetilde{\boldsymbol{\mu}},\widetilde{\Sigma}}(\widetilde{\mathbf{x}},\widetilde{\mathbf{u}})|$. Using (5.4.4) and the RHS of (5.4.16), we have

$$\begin{aligned}
|\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{x},\mathbf{u})| &= |\langle \mathbf{x}, \mathbf{u}\diamond\boldsymbol{\mu}\rangle_{(\mathbf{u}\diamond\Sigma)^{-1}}| \\
&\leq |\mathfrak{E}_{\widetilde{\boldsymbol{\mu}},\widetilde{\Sigma}}(\widetilde{\mathbf{x}},\widetilde{\mathbf{u}})| + D(\Sigma)|\mathfrak{E}_{\widetilde{\boldsymbol{\sigma}},\widetilde{\Sigma}}(\widetilde{\mathbf{x}},\widetilde{\mathbf{u}})\mathfrak{E}_{\widetilde{\boldsymbol{\sigma}},\widetilde{\Sigma}}(\widetilde{\mathbf{u}}\diamond\widetilde{\boldsymbol{\mu}},\widetilde{\mathbf{u}}) \\
&\qquad\qquad - u\mu\mathfrak{E}_{\widetilde{\boldsymbol{\sigma}},\widetilde{\Sigma}}(\widetilde{\mathbf{x}},\widetilde{\mathbf{u}}) - x\mathfrak{E}_{\widetilde{\boldsymbol{\sigma}},\widetilde{\Sigma}}(\widetilde{\mathbf{u}}\diamond\widetilde{\boldsymbol{\mu}},\widetilde{\mathbf{u}}) + xu\mu| \\
&\leq \overline{\mathfrak{E}}_{\widetilde{\boldsymbol{\mu}},\widetilde{\mathbf{x}}} + D(\Sigma)(\overline{\mathfrak{E}}_{\widetilde{\boldsymbol{\sigma}},\widetilde{\mathbf{x}}}\xi_{\widetilde{\boldsymbol{\mu}}}^{1/2}\xi_{\widetilde{\boldsymbol{\sigma}}}^{1/2} + |\mu|\overline{\mathfrak{E}}_{\widetilde{\boldsymbol{\sigma}},\widetilde{\mathbf{x}}} + |x|\xi_{\widetilde{\boldsymbol{\mu}}}^{1/2}\xi_{\widetilde{\boldsymbol{\sigma}}}^{1/2} + |x\mu|),
\end{aligned}$$

where the last line follows by using (5.4.22) and $u < 1$. This is finite by the inductive hypothesis. Thus, $\overline{\mathfrak{E}}_{\boldsymbol{\mu},\mathbf{x}} < \infty$. □

Remark 5.4.5. For $\mathbf{y} \in \mathbb{R}^n_*$, it is possible that $\sup_{\mathbf{u}\in\mathbb{S}_{++}} \|\mathbf{y}\|_{(\mathbf{u}\diamond\Sigma)^{-1}} = \infty$ and $\inf_{\mathbf{u}\in\mathbb{S}_{++}} \mathfrak{D}_\Sigma(\mathbf{y},\mathbf{u}) = 0$. For example, take $\Sigma = I_2 \in \mathbb{R}^{2\times 2}$ to be the identity matrix and $\mathbf{y} = \mathbf{e}_1 \in \mathbb{R}^2$ to see this.

For $\mathbf{x} = (x_1,\dots,x_n) \in \mathbb{R}^n$, let

$$\min_{1\leq k\leq n}{}^{*}\, x_k := \min\{x_k : x_k \neq 0,\ 1 \leq k \leq n\}$$

with the convention $\min\emptyset = -\infty$. Introduce a family of compact neighbourhoods of $\mathbf{0}$ in $\mathbb{R}^n$ by setting

$$\mathbb{A}(\mathbf{y}) := \left(\min_{1\leq k\leq n}{}^{*}\, |y_k|\right)[-1,1]^n,$$

indexed by $\mathbf{y} = (y_1,\dots,y_n) \in \mathbb{R}^n_*$.

Recall that Σ is an invertible covariance matrix. Next, we state uniform versions of the bounds in Lemma 5.4.4.

Lemma 5.4.6.

(i) If $\boldsymbol{\mu}, \mathbf{w} \in \mathbb{R}^n$, *then* $\sup_{\mathbf{x}\in\mathbf{w}+\mathbb{D}} \sup_{\mathbf{u}\in\mathbb{S}_{++}} |\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{x},\mathbf{u})| < \infty$.

(ii) If $\mathbf{y} \in \mathbb{R}^n_*$, *then there exist* $\epsilon > 0$ *such that* $\inf_{\mathbf{x}\in\mathbf{y}+\epsilon\mathbb{A}(\mathbf{y})} \inf_{\mathbf{u}\in\mathbb{S}_{++}} \|\mathbf{x}\|_{(\mathbf{u}\diamond\Sigma)^{-1}} > 0$.

Proof. (i). Let $\boldsymbol{\mu}, \mathbf{w}, \mathbf{x} \in \mathbb{R}^n$ and $\mathbf{u} \in (0,\infty)^n$. Recalling the definition of $\mathfrak{E}_{\boldsymbol{\mu},\Sigma}$ in (5.3.6) and applying the triangle inequality yields

$$|\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{x},\mathbf{u})| \leq |\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{w},\mathbf{u})| + |\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{x}-\mathbf{w},\mathbf{u})|$$

Note that $|\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{x}-\mathbf{w},\mathbf{u})| = |\langle \mathbf{x}-\mathbf{w}, (\mathbf{u}\diamond\boldsymbol{\mu})(\mathbf{u}\diamond\Sigma)^{-1}\rangle|$, so using the Cauchy-Schwarz inequality gives

$$\begin{aligned}|\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{x}-\mathbf{w},\mathbf{u})| &\leq \|\mathbf{x}-\mathbf{w}\|\|(\mathbf{u}\diamond\boldsymbol{\mu})(\mathbf{u}\diamond\Sigma)^{-1}\| \\ &= \|\mathbf{x}-\mathbf{w}\|\left(\sum_{k=1}^{n}\mathfrak{E}^2_{\boldsymbol{\mu},\Sigma}(\mathbf{e}_k,\mathbf{u})\right)^{1/2}. \end{aligned} \tag{5.4.23}$$

Combining the above results yields

$$|\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{x},\mathbf{u})| \leq |\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{w},\mathbf{u})| + \|\mathbf{x}-\mathbf{w}\|\left(\sum_{k=1}^{n}\mathfrak{E}^2_{\boldsymbol{\mu},\Sigma}(\mathbf{e}_k,\mathbf{u})\right)^{1/2}.$$

Therefore, the finiteness of the iterated supremum is implied by (5.4.14).

(ii). Let $n \geq 2$. Let $\widetilde{\mathbf{e}}_1,\ldots,\widetilde{\mathbf{e}}_{n-1}$ be the canonical basis vectors of $\mathbb{R}^{n-1}$, and $\mathbb{P}_n$ be the set of all $n\times n$ permutation matrices. For $P \in \mathbb{P}_n$, write

$$\Sigma_P := P'\Sigma P = \begin{pmatrix} \widetilde{\Sigma}_P & \widetilde{\boldsymbol{\sigma}}'_P \\ \widetilde{\boldsymbol{\sigma}}_P & \sigma_P \end{pmatrix},$$

where $\widetilde{\Sigma}_P \in \mathbb{R}^{(n-1)\times(n-1)}$, $\widetilde{\boldsymbol{\sigma}}_P \in \mathbb{R}^{n-1}$ and $\sigma_P \in (0,\infty)$. Define

$$M_n(\Sigma) := 2\sum_{l=1}^{n-1}\max_{P\in\mathbb{P}_n}\sup_{\widetilde{\mathbf{u}}\in\mathbb{S}_{++}}|\mathfrak{E}_{\widetilde{\boldsymbol{\sigma}}_P,\widetilde{\Sigma}_P}(\widetilde{\mathbf{e}}_l,\widetilde{\mathbf{u}})|. \tag{5.4.24}$$

By (5.4.14), $M_n(\Sigma) \in [0,\infty)$.

Let $\mathbf{y} = (y_1,\ldots,y_n) \in \mathbb{R}^n_*$. We show by mathematical induction that there exist $0 < \epsilon < 1$, $E_n(\Sigma) \in (0,\infty)$ and $F_n(\Sigma) \in [0,\infty)$ such that

$$\|\mathbf{x}\|^2_{(\mathbf{u}\diamond\Sigma)^{-1}} \geq E_n(\Sigma)((1-\epsilon)^2 - \epsilon(1+\epsilon)F_n(\Sigma))\min_{1\leq k\leq n}{}^{*}\, y_k^2, \tag{5.4.25}$$

$$(1-\epsilon)^2 - \epsilon(1+\epsilon)F_n(\Sigma) > 0 \tag{5.4.26}$$

for all $\mathbf{x} \in \mathbf{y} + \epsilon\mathbb{A}(\mathbf{y})$ and $\mathbf{u} \in \mathbb{S}_{++}$. If this holds, it implies the iterated infimum is strictly positive, completing the proof of Part (ii).

If $n = 1$, note that $y \neq 0$, $u = 1$, and $x \in y + \epsilon\mathbb{A}(y) = [y - \epsilon|y|, y + \epsilon|y|]$, which is the ball centred at y with radius $\epsilon|y|$, $0 < \epsilon < 1$. Considering the cases $y < 0$ and $y > 0$, this clearly implies

$$0 < (1-\epsilon)|y| \leq |x| \leq (1+\epsilon)|y|. \tag{5.4.27}$$

In particular, $\|x\|^2_{(u\diamond\Sigma)^{-1}} = x^2/\Sigma \geq (1-\epsilon)^2 y^2/\Sigma$ for $x \in y + \epsilon\mathbb{A}(y)$. Thus, the assertion holds for $E_1(\Sigma) = 1/\Sigma$, $F_1(\Sigma) = 0$.

Next, assume $n \geq 2$ and the inductive hypothesis holds for $n-1$. We can assume without loss of generality that $\mathbf{u} \in \mathbb{S} \cap (0,\infty)^n_{\leq}$ by (5.4.2). We use dimension reduction notation for $\mathbf{u}$, $\mathbf{x}$, $\mathbf{y}$, Σ. For $1 \leq k \leq n$, let $\Sigma^{(k)} \in \mathbb{R}^{(n-1)\times(n-1)}$ be the invertible covariance matrix obtained from Σ by deleting its kth row and column. Set $\widetilde{E}_n(\Sigma) := \min_{1\leq k\leq n} E_{n-1}(\Sigma^{(k)})$ and $\widetilde{F}_n(\Sigma) := \max_{1\leq k\leq n} F_{n-1}(\Sigma^{(k)})$.

First, assume $\widetilde{\mathbf{y}} \neq \mathbf{0}$. Recall that $\widetilde{\mathbf{u}}^0 = \widetilde{\mathbf{u}}/\|\widetilde{\mathbf{u}}\|$. If $\mathbf{x} \in \mathbf{y} + \epsilon\mathbb{A}(\mathbf{y})$, then $\widetilde{\mathbf{x}} \in \widetilde{\mathbf{y}} + \epsilon\mathbb{A}(\widetilde{\mathbf{y}})$. Thus, we have

$$\begin{aligned}\|\mathbf{x}\|^2_{(\mathbf{u}\diamond\Sigma)^{-1}} \geq \|\widetilde{\mathbf{x}}\|^2_{(\widetilde{\mathbf{u}}^0\diamond\widetilde{\Sigma})^{-1}} &\geq E_{n-1}(\widetilde{\Sigma})((1-\epsilon)^2 - \epsilon(1+\epsilon)F_{n-1}(\widetilde{\Sigma})) \min_{1\leq k\leq n-1}{}^{*}\, y_k^2 \\ &\geq \widetilde{E}_n(\Sigma)((1-\epsilon)^2 - \epsilon(1+\epsilon)\widetilde{F}_n(\Sigma)) \min_{1\leq k\leq n}{}^{*}\, y_k^2,\end{aligned}$$

where we can use (5.4.19), since $\mathbf{x} \neq \mathbf{0}$, to obtain the first inequality, and the inductive hypothesis to obtain the second inequality. Also, by the inductive hypothesis, $(1-\epsilon)^2 - \epsilon(1+\epsilon)\widetilde{F}_n(\Sigma) > 0$.

Second, assume $\widetilde{\mathbf{y}} = \mathbf{0}$, so $y \neq 0$. If $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{y} + \epsilon\mathbb{A}(\mathbf{y}) = (\mathbf{0}, y) + \epsilon|y|[-1,1]^n$, $0 < \epsilon < 1$, then $\max_{1\leq k\leq n-1} |x_k| \leq \epsilon|y|$, and (5.4.27) also holds. Thus, it follows from the definition of $\mathfrak{C}_{\widetilde{\boldsymbol{\sigma}},\widetilde{\Sigma}}$ in (5.3.6) that

$$\begin{aligned}2|\mathfrak{C}_{\widetilde{\boldsymbol{\sigma}},\widetilde{\Sigma}}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{u}})| &= 2\left|\sum_{l=1}^{n-1} x_k \mathfrak{C}_{\widetilde{\boldsymbol{\sigma}},\widetilde{\Sigma}}(\widetilde{\mathbf{e}}_l, \widetilde{\mathbf{u}})\right| \\ &\leq \left(\max_{1\leq k\leq n-1} |x_k|\right) M_n(\Sigma) \\ &\leq M_n(\Sigma)\epsilon|y|, \end{aligned} \tag{5.4.28}$$

with $M_n(\Sigma)$ from (5.4.24). Recall the definition of $C(\Sigma)$ from (5.4.15), and set $\widetilde{C}_n(\Sigma) := \min_{1\leq k\leq n} |\Sigma^{(k)}| / \prod_{k=1}^n \Sigma_{kk}$. Combining (5.4.17), which is applicable as $\mathbf{x} \neq \mathbf{0}$, with $(a-b)^2 \geq |a|^2 - 2|a||b|$, $a, b \in \mathbb{R}$, which is implied by the reverse triangle inequality, we obtain

$$\begin{aligned}\|\mathbf{x}\|^2_{(\mathbf{u}\diamond\Sigma)^{-1}} &\geq C(\Sigma)(x - \mathfrak{C}_{\widetilde{\boldsymbol{\sigma}},\widetilde{\Sigma}}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{u}}))^2 \\ &\geq C(\Sigma)(|x|^2 - 2|x||\mathfrak{C}_{\widetilde{\boldsymbol{\sigma}},\widetilde{\Sigma}}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{u}})|) \\ &\geq C(\Sigma)((1-\epsilon)^2 y^2 - \epsilon(1+\epsilon)M_n(\Sigma)y^2) \\ &\geq \widetilde{C}_n(\Sigma)((1-\epsilon)^2 - \epsilon(1+\epsilon)M_n(\Sigma)) \min_{1\leq k\leq n}{}^{*}\, y_k^2,\end{aligned}$$

where the second last line follows by using (5.4.27) and (5.4.28). In addition,

$\epsilon(1+\epsilon)M_n(\Sigma) < (1-\epsilon)^2$ for some sufficiently small $0 < \epsilon < 1$.

To summarise, assuming the inductive hypothesis, there exists $0 < \epsilon < 1$, $E_n(\Sigma) := \widetilde{E}_n(\Sigma) \wedge \widetilde{C}_n(\Sigma)$ and $F_n(\Sigma) := \widetilde{F}_n(\Sigma) \vee M_n(\Sigma)$ such that (5.4.25) and (5.4.26) hold for all $\mathbf{y} \in \mathbb{R}^n_*$, $\mathbf{x} \in \mathbf{y} + \epsilon\mathbb{A}(\mathbf{y})$ and $\mathbf{u} \in \mathbb{S}_{++}$. This completes the proof by mathematical induction. ☐

5.4.2 Uniform Positivity of $\mathfrak{E}$

Here, we prove the uniform positivity of $\mathfrak{E}_{\boldsymbol{\mu},\Sigma}$, in the sense that $\inf_{\mathbf{u}\in(0,\infty)^n} \mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{x}, \mathbf{u}) > 0$ for $\mathbf{x}$ in some open set containing $\boldsymbol{\mu} \neq \mathbf{0}$. This result, stated in Proposition 5.4.11, is important to proving Theorem 5.3.3 (ii).

We can see that this result straightforwardly holds in some limited cases. Recall that $\Sigma \in \mathbb{R}^{n\times n}$ is an invertible covariance matrix. For $n = 1$, $\mathfrak{E}_{\mu,\Sigma}(\mu, u) = \mu^2/\Sigma > 0$, $u > 0$. For $VGG^{n,1}$ processes, $\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\boldsymbol{\mu}, u\mathbf{e}) = \|\boldsymbol{\mu}\|^2_{\Sigma^{-1}} > 0$, $u > 0$. For $VGG^{n,n}$ processes, $\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\boldsymbol{\mu}, \mathbf{u}) = \|\boldsymbol{\mu}\|^2_{\Sigma^{-1}} > 0$, $\mathbf{u} \in (0,\infty)^n$. In each of these cases uniform positivity holds. However, for $WVAG^n$ processes, $\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\boldsymbol{\mu}, g\boldsymbol{\alpha}) = \langle \boldsymbol{\mu}, \boldsymbol{\alpha} \diamond \boldsymbol{\mu}\rangle_{(\boldsymbol{\alpha}\diamond\Sigma)^{-1}}$, $g > 0$, and determining whether this is positive is non-trivial.

We begin by proving some lemmas. If $n = 1$, set $\Xi_n(x) \equiv 2$, $x \in \mathbb{R}$. Otherwise, if $n \geq 2$, let $\Xi_n(\mathbf{x}) = (\Xi_{n,kl}) \in \mathbb{R}^{n\times n}$, $\mathbf{x} = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, be defined by

$$\Xi_{n,kl}(\mathbf{x}) := \begin{cases} 2 & \text{if } k = l, \\ x_k & \text{if } 1 \leq k < n \text{ and } l \neq k, \\ 1 & \text{if } k = n \text{ and } l \leq l < n. \end{cases} \tag{5.4.29}$$

Recall that ∂A denotes the boundary of $A \subseteq \mathbb{R}^n$ relative to $\mathbb{R}^n$.

Lemma 5.4.7. *For $n \geq 1$, $\inf_{\mathbf{x}\in[0,1]^n} |\Xi_{n+1}(\mathbf{x})| = 2 + n$ and $\sup_{\mathbf{x}\in[0,1]^n} |\Xi_{n+1}(\mathbf{x})| = 2^{n+1}$.*

Proof. If $n = 1$, then the result is trivial. Otherwise, assume $n \geq 2$.

Let $h_n(\mathbf{x}) := |\Xi_{n+1}(\mathbf{x})|$, $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, which is a polynomial of degree n in the variables $x_1, \dots, x_n$. Expanding the determinant along its first row yields $h_n(\mathbf{x}) = 2h_{n-1}(\widetilde{\mathbf{x}}) + x_1 r_{n-1}(\widetilde{\mathbf{x}})$, $\mathbf{x} = (x_1, \widetilde{\mathbf{x}})$, where $x_1 \in \mathbb{R}$, $\widetilde{\mathbf{x}} = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$, and $\widetilde{\mathbf{x}} \mapsto r_{n-1}(\widetilde{\mathbf{x}})$ is a remainder polynomial .

Note that $x_1 \mapsto h_n(x_1, \widetilde{\mathbf{x}})$ is an affine function in its first variable, so that $\partial^2_{x_1} h_n(x_1, \widetilde{\mathbf{x}}) \equiv 0$. Also, h_n is invariant under coordinate permutations so that $h_n(\mathbf{x}P) = h_n(\mathbf{x})$ for any permutation matrix $P \in \mathbb{R}^{n\times n}$. Thus, h_n is a harmonic function, meaning that $\mathrm{div}(h_n) \equiv 0$.

The maximum principle for harmonic functions (see Section 1.I.4 in [Doo01]) states that

$$\inf_{\mathbf{x}\in[0,1]^n} h_n(\mathbf{x}) = \min_{\mathbf{x}\in\partial[0,1]^n} h_n(\mathbf{x}), \qquad \sup_{\mathbf{x}\in[0,1]^n} h_n(\mathbf{x}) = \max_{\mathbf{x}\in\partial[0,1]^n} h_n(\mathbf{x}).$$

Due to the permutation invariance, to determine the infimum and supremum of h_n, we only need to check its value on the boundary points

$$\mathbb{F}_n := \{\mathbf{0},\mathbf{e}\} \cup \bigcup_{1\le k<n} \{\mathbf{x} = (x_1,\dots,x_n) : x_1 = \cdots = x_k = 0,\, x_{k+1} = \cdots = x_n = 1\}.$$

Obviously, $h_n(\mathbf{0}) = 2^{n+1}$, while $h_n(\mathbf{e}) = 2+n$ (see Theorem 8.4.4 in [Gra83]). For $1 \le k < n$, we have $h_n(\mathbf{x}) = 2^k h_{n-k}(1,\dots,1) = 2^k(2+n-k)$ when $\mathbf{x} = (x_1,\dots,x_n)$, $x_1 = \cdots = x_k = 0$, $x_{k+1} = \cdots = x_n = 1$. Thus,

$$\inf_{\mathbf{x}\in[0,1]^n} h_n(\mathbf{x}) = \min_{\mathbf{x}\in\mathbb{F}_n} h_n(\mathbf{x}) = 2+n, \qquad \sup_{\mathbf{x}\in[0,1]^n} h_n(\mathbf{x}) = \max_{\mathbf{x}\in\mathbb{F}_n} h_n(\mathbf{x}) = 2^{n+1},$$

which completes the proof. □

Remark 5.4.8. The matrix $\Xi_{n+1}(\mathbf{e})$ is the covariance matrix of some $n+1$ equicorrelated random variables.

For $n \ge 2$ and $\mathbf{w} = (w_1,\dots w_{n-1}) \in \mathbb{R}^{n-1}$, define the symmetric matrix $\Upsilon_n(\mathbf{w}) := (\Upsilon_{n,kl}(\mathbf{w})) \in \mathbb{R}^{n\times n}$ by setting

$$\begin{aligned}\Upsilon_{n,kl}(\mathbf{w}) &:= \begin{cases} 2 & \text{if } k = l, \\ 1 + \prod_{k\le m<l} w_m & \text{if } 1 \le k < l \le n, \end{cases} \\ \Upsilon_{n,lk}(\mathbf{w}) &:= \Upsilon_{n,kl}(\mathbf{w}), \quad 1 \le k < l \le n.\end{aligned} \tag{5.4.30}$$

Lemma 5.4.9. *If $n \ge 2$ and $\mathbf{w} \in [0,1]^{n-1}$, then $\Upsilon_n(\mathbf{w})$ is nonnegative definite.*

Proof. We perform the following three operations on $\Upsilon_n(\mathbf{w})$ for $k = 1$ to $k = n-1$. All three operations are completed before moving to the next iteration of k. Firstly, multiply its $(k+1)$th column by w_k and subtract this from its kth column. Secondly, multiply its $(k+1)$th row by w_k and subtract this from its kth row. Thirdly, factor out $x_k := 1 - w_k$ from the kth column if $x_k \in (0,1]$.

If $x_k \in (0,1]$ for all $1 \le k < n$, this yields $|\Upsilon_n(\mathbf{w})| = |\Xi_n(\mathbf{x})| \prod_{1\le k<n} x_k$, where $\Xi_n(\mathbf{x}) \in \mathbb{R}^{n\times n}$ is defined in (5.4.29). Thus, by Lemma 5.4.7, we have $|\Upsilon_n(\mathbf{w})| > 0$, $\mathbf{w} \in [0,1]^{n-1}$. Otherwise, if there exists some $x_k = 0$ for $1 \le k < n$, then the kth column is zero, so that $|\Upsilon_n(\mathbf{w})| = 0$, $\mathbf{w} \in [0,1]^{n-1}$.

Every other principal submatrix of $\Upsilon_n(\mathbf{w})$, formed by keeping the rows and columns in the index set $\{j_1, \dots j_m\}$, $1 \le j_1 < \dots < j_m \le n$, $1 \le m \le n-1$ and deleting the rest, is 2 if $m = 1$, otherwise it is given by $\Upsilon_m(\overline{\mathbf{w}})$, where

$$\overline{\mathbf{w}} := \left(\prod_{k=j_1}^{j_2-1} w_k, \dots, \prod_{k=j_{m-1}}^{j_m-1} w_k \right) \in [0,1]^{m-1}.$$

Repeating the above argument on $\Upsilon_m(\overline{\mathbf{w}})$ shows that $|\Upsilon_m(\overline{\mathbf{w}})| \ge 0$. Hence, $\Upsilon_n(\mathbf{w})$ is nonnegative definite for all $\mathbf{w} \in [0,1]^{n-1}$ by Sylvester's criterion. $\square$

For $n \ge 2$ and $\mathbf{w} = (w_1, \dots, w_{n-1}) \in \mathbb{R}^{n-1}$, define the matrix $\Delta_n(\mathbf{w}) := (\Delta_{n,kl}(\mathbf{w})) \in \mathbb{R}^{n \times n}$ by setting

$$\Delta_{n,kl}(\mathbf{w}) := \begin{cases} \prod_{k \le m < l} w_m & \text{if } 1 \le k < l \le n, \\ 1 & \text{if } 1 \le l \le k \le n. \end{cases} \tag{5.4.31}$$

Recall that $\Sigma = (\Sigma_{kl}) \in \mathbb{R}^{n \times n}$ an invertible covariance matrix and $*$ denotes the Hadamard product.

Lemma 5.4.10. *If $n \ge 2$ and $\mathbf{w} \in [0,1]^{n-1}$, then $\Delta_n(\mathbf{w}) * \Sigma$ is invertible.*

Proof. Recall that $(0,\infty)^n_{\le} := \{\mathbf{u} = (u_1, \dots, u_n) \in (0,\infty)^n : u_1 \le \dots \le u_n\}$. The mapping

$$\mathbf{u} = (u_1, \dots, u_n) \mapsto \mathbf{w} = (u_1/u_2, \dots, u_{n-1}/u_n) \tag{5.4.32}$$

defines a bijection from $\mathbb{S} \cap (0,\infty)^n_{\le}$ to $(0,1]^{n-1}$ with inverse $\mathbf{w}^{-1} : (0,1]^{n-1} \to \mathbb{S} \cap (0,\infty)^n_{\le}$. Note that $\Delta_n(\mathbf{w}) * \Sigma = (\mathbf{u} \diamond \Sigma) \operatorname{diag}(1/\mathbf{u})$, where $\operatorname{diag}(1/\mathbf{u}) := \operatorname{diag}(1/u_1, \dots, 1/u_n)$. Thus, $\Delta_n(\mathbf{w}) * \Sigma$ is invertible for all $\mathbf{w} \in (0,1]^{n-1}$ because $(\mathbf{u} \diamond \Sigma) \operatorname{diag}(1/\mathbf{u})$ is invertible due to (5.4.1).

It remains to show invertibility for $\mathbf{w} \in \mathbb{E}_n$, where

$$\mathbb{E}_n := \bigcup_{k=1}^{n-1} \{\mathbf{w} = (w_1, \dots, w_{n-1}) \in \partial[0,1]^{n-1} : w_k = 0\}. \tag{5.4.33}$$

We use mathematical induction. If $n = 2$, note that $\mathbb{E}_n = \{0\}$, so that

$$\Delta_2(w) * \Sigma = \Delta_2(0) * \Sigma = \begin{pmatrix} \Sigma_{11} & 0 \\ \Sigma_{12}, & \Sigma_{22} \end{pmatrix},$$

which is invertible as the product of the diagonal is strictly positive. Next, assume $n \geq 3$ and $\mathbf{w} \in \mathbb{E}_n$. If $w_1 = 0$, then,

$$\Delta_n(\mathbf{w}) * \Sigma = \begin{pmatrix} \Sigma_{11} & \mathbf{0} \\ \widetilde{\boldsymbol{\sigma}}' & \Delta_{n-1}(w_2, \ldots, w_{n-1}) * \widetilde{\Sigma} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \widetilde{\boldsymbol{\sigma}} \\ \widetilde{\boldsymbol{\sigma}}' & \widetilde{\Sigma} \end{pmatrix}.$$

If $w_{n-1} = 0$, then

$$\Delta_n(\mathbf{w}) * \Sigma = \begin{pmatrix} \Delta_{n-1}(w_1, \ldots, w_{n-2}) * \widetilde{\Sigma} & \mathbf{0}' \\ \widetilde{\boldsymbol{\sigma}} & \Sigma_{nn} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \widetilde{\Sigma} & \widetilde{\boldsymbol{\sigma}}' \\ \widetilde{\boldsymbol{\sigma}} & \Sigma_{nn} \end{pmatrix}.$$

Otherwise, there exists a $1 < k < n-1$ such that $w_k = 0$, and we have

$$\Delta_n(\mathbf{w}) * \Sigma = \begin{pmatrix} \Delta_k(w_1, \ldots, w_{k-1}) * \widetilde{\Sigma}_{11} & 0 \\ \widetilde{\Sigma}_{21} & \Delta_{n-k}(w_{k+1}, \ldots, w_{n-1}) * \widetilde{\Sigma}_{22} \end{pmatrix},$$
$$\Sigma = \begin{pmatrix} \widetilde{\Sigma}_{11} & \widetilde{\Sigma}_{12} \\ \widetilde{\Sigma}_{21} & \widetilde{\Sigma}_{22} \end{pmatrix},$$

where $\widetilde{\Sigma}_{11} \in \mathbb{R}^{k \times k}, \widetilde{\Sigma}_{22} \in \mathbb{R}^{(n-k)\times(n-k)}$. In all these cases, $\Delta_n(\mathbf{w}) * \Sigma$ is invertible as the product of the determinants of the block diagonal is strictly positive due to the inductive hypothesis and Σ being positive definite. This completes the proof. $\square$

Now we introduce the set of points $\mathbb{V}^+_{\boldsymbol{\mu},\Sigma}$ where uniform positivity of $\mathfrak{E}_{\boldsymbol{\mu},\Sigma}$ holds. Let

$$\mathbb{V}^+_{\boldsymbol{\mu},\Sigma} := \left\{ \mathbf{x} \in \mathbb{R}^n : \inf_{\mathbf{u} \in (0,\infty)^n} \mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{x}, \mathbf{u}) > 0 \right\}. \tag{5.4.34}$$

We show that this set is nonempty and open.

For a matrix $A \in \mathbb{R}^{n \times n}$, let $\operatorname{sym}(A) := (A + A')/2 \in \mathbb{R}^{n \times n}$ denote the symmetrisation of A. In particular, $\operatorname{sym}(A)$ is always a symmetric matrix.

Proposition 5.4.11. *Let $\boldsymbol{\mu} \in \mathbb{R}^n$. The set $\mathbb{V}^+_{\boldsymbol{\mu},\Sigma}$ is an open convex cone of $\mathbb{R}^n$. If $\boldsymbol{\mu} \neq \mathbf{0}$, then $\boldsymbol{\mu} \in \mathbb{V}^+_{\boldsymbol{\mu},\Sigma} \neq \emptyset$.*

Proof. Open convex cone. If $a, b > 0$ and $\mathbf{x}, \mathbf{y} \in \mathbb{V}^+_{\boldsymbol{\mu},\Sigma}$, then it immediately follows that $a\mathbf{x} + b\mathbf{y} \in \mathbb{V}^+_{\boldsymbol{\mu},\Sigma}$, so $\mathbb{V}^+_{\boldsymbol{\mu},\Sigma}$ is a convex cone.

For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{u} \in (0,\infty)^n$, (5.4.23) gives

$$|\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{x}, \mathbf{u}) - \mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{y}, \mathbf{u})| \leq C \|\mathbf{x} - \mathbf{y}\|, \quad C := 1 + \left(\sum_{k=1}^{n} \sup_{\mathbf{u} \in (0,\infty)^n} \mathfrak{E}^2_{\boldsymbol{\mu},\Sigma}(\mathbf{e}_k, \mathbf{u}) \right)^{1/2},$$

where C is a finite constant due to (5.4.14), and it is positive due to the addition of 1. Now choose $\mathbf{x} \in \mathbb{V}^+_{\boldsymbol{\mu},\Sigma}$ so that $\underline{\mathfrak{E}} := \inf_{\mathbf{u}\in(0,\infty)^n} \mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{x},\mathbf{u}) > 0$, and choose $\mathbf{y} \in \mathbb{R}^n$ satisfying $\|\mathbf{x}-\mathbf{y}\| \leq \underline{\mathfrak{E}}/(2C)$. With these choices, we have

$$\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{y},\mathbf{u}) \geq \mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{x},\mathbf{u}) - C\|\mathbf{x}-\mathbf{y}\| \geq \frac{\underline{\mathfrak{E}}}{2}.$$

Thus, $\inf_{\mathbf{u}\in(0,\infty)^n} \mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{y},\mathbf{u}) \geq \underline{\mathfrak{E}}/2$, so $\mathbf{y} \in \mathbb{V}^+_{\boldsymbol{\mu},\Sigma}$. This shows that $\mathbb{V}^+_{\boldsymbol{\mu},\Sigma}$ is open.

Positivity. For the remainder of the proof, assume that $\boldsymbol{\mu} \neq \mathbf{0}$. We show that $\mathfrak{E}_{\mu,\Sigma}(\boldsymbol{\mu},\mathbf{u}) > 0$, $\mathbf{u} \in (0,\infty)^n$. Recall that $\mathbb{S}_{++} := \mathbb{S} \cap (0,\infty)^n$. By (5.3.8), $\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{x},\mathbf{u}) = \mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\mathbf{x},\mathbf{u}^0)$, where $\mathbf{u}^0 := \mathbf{u}/\|\mathbf{u}\|$ and $\mathbf{u} \in (0,\infty)^n$, so we assume without loss of generality that $\mathbf{u} \in \mathbb{S}_{++}$.

Let $\Sigma^s(\mathbf{u},\Sigma) := \operatorname{sym}((\mathbf{u}\diamond\Sigma)\operatorname{diag}(1/\mathbf{u}))$, where $\operatorname{diag}(1/\mathbf{u}) := \operatorname{diag}(1/u_1,\ldots,1/u_n)$. We have

$$\begin{aligned}
\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\boldsymbol{\mu},\mathbf{u}) &= \boldsymbol{\mu}(\mathbf{u}\diamond\Sigma)^{-1}\operatorname{diag}(\mathbf{u})\boldsymbol{\mu}' &(5.4.35)\\
&= \|\boldsymbol{\mu}((\mathbf{u}\diamond\Sigma)\operatorname{diag}(1/\mathbf{u}))^{-1}\|^2_{(\mathbf{u}\diamond\Sigma)\operatorname{diag}(1/\mathbf{u})} \\
&= \|\boldsymbol{\mu}((\mathbf{u}\diamond\Sigma)\operatorname{diag}(1/\mathbf{u}))^{-1}\|^2_{\Sigma^s(\mathbf{u},\Sigma)}, &(5.4.36)
\end{aligned}$$

where the last line follows from Lemma A.2.3. Thus, $\mathfrak{E}_{\mu,\Sigma}(\boldsymbol{\mu},\mathbf{u}) > 0$ provided that $\Sigma^s(\mathbf{u},\Sigma)$ is positive definite.

Introduce $g(u) := (1\wedge u) + (1\wedge(1/u)) \in (1,2]$, $u > 0$, and the symmetric matrix $\Theta_n(\mathbf{u}) := (\Theta_{n,kl}(\mathbf{u}))$ defined by $\Theta_{n,kl}(\mathbf{u}) := g(u_k/u_l)$, $1 \leq k,l \leq n$, $\mathbf{u} \in \mathbb{S}_{++}$. We now prove that $\Theta_n(\mathbf{u})$ is nonnegative definite. For $n = 1$, this is clear as $|\Theta_1(u)| \equiv 2$.

For $n \geq 2$, note that $\Theta_n(\mathbf{u}P) = P'\Theta_n(\mathbf{u})P$, $\mathbf{u} \in \mathbb{S}_{++}$, for any permutation matrix $P \in \mathbb{R}^{n\times n}$ as the (k,l) elements of both sides are $g(u_{(k)}/u_{(l)})$, where $\langle(1),\ldots,(n)\rangle$ is the permutation associated with P. Thus, we can assume $\mathbf{u} \in \mathbb{S}\cap(0,\infty)^n_{\leq}$ without loss of generality as proving that $\Theta_n(\mathbf{u})$ is nonnegative definite for $\mathbf{u} \in \mathbb{S}\cap(0,\infty)^n_{\leq}$ implies it for all $\mathbf{u} \in \mathbb{S}_{++}$. This follows from the definition of a nonnegative definite matrix. Now with $\mathbf{w} \in (0,1]^{n-1}$ determined by the bijection in (5.4.32), we have $\Theta_n(\mathbf{u}) = \Upsilon_n(\mathbf{w})$, where $\Upsilon_n(\mathbf{w})$ is defined in (5.4.30) and nonnegative definite by Lemma 5.4.9.

Finally, note that $2\Sigma^s(\mathbf{u},\Sigma) = \Theta_n(\mathbf{u}) * \Sigma$, $\mathbf{u} \in \mathbb{S}_{++}$. Since Σ is positive definite by assumption and $\Theta_n(\mathbf{u})$ is nonnegative definite with positive diagonal elements $\Theta_{n,kk}(\mathbf{u}) \equiv 2$, $1 \leq k \leq n$, Oppenheim's inequality (see the LHS of (A.2.1)) immediately implies that every leading principal minor of $\Theta_n(\mathbf{u}) * \Sigma$ is positive. Thus, $\Sigma^s(\mathbf{u},\Sigma)$ is positive definite, proving that $\mathfrak{E}_{\mu,\Sigma}(\boldsymbol{\mu},\mathbf{u}) > 0$, $\mathbf{u} \in (0,\infty)^n$.

Uniform positivity. Now we show that $\boldsymbol{\mu} \in \mathbb{V}^+_{\boldsymbol{\mu},\Sigma}$. If $n = 1$, the result is obvious as $\mathfrak{E}_{\mu,\Sigma}(\mu, u) \equiv \mu^2/\Sigma$. Now assume $n \geq 2$. Let $\underline{\mathfrak{E}} := \inf_{\mathbf{u}\in(0,\infty)^n} \mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\boldsymbol{\mu}, \mathbf{u})$ and note that $\underline{\mathfrak{E}} = \inf_{\mathbf{u}\in\mathbb{S}_{++}} \mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\boldsymbol{\mu}, \mathbf{u})$ by (5.3.8).

Clearly, there exists a sequence $(\mathbf{u}_m)_{m\in\mathbb{N}} \subseteq \mathbb{S}_{++}$ such that $\lim_{m\to\infty} \mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\boldsymbol{\mu}, \mathbf{u}_m) = \underline{\mathfrak{E}}$. Without loss of generality, by choosing a suitable subsequence if necessary, we may assume that $\mathbf{u}_m \to \mathbf{u}_0$ for some $\mathbf{u}_0 \in \mathbb{S}_+$ as $m \to \infty$, where $\mathbb{S}_+ := \mathbb{S} \cap [0,\infty)^n$.

If $\mathbf{u}_0 \in \mathbb{S}_{++}$, then $\underline{\mathfrak{E}} = \mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\boldsymbol{\mu}, \mathbf{u}_0) > 0$ follows from the positivity result we have proven. Otherwise, assume that $\mathbf{u}_0 \in \mathbb{S}_+\backslash\mathbb{S}_{++}$. Without loss of generality, we can further assume that $(\mathbf{u}_m)_{m\in\mathbb{N}} \subseteq (0,\infty)^n_{\leq}$ due to (5.4.2). By selecting a suitable subsequence if necessary, and with the bijection in (5.4.32), we may assume that $\mathbf{w}_m := \mathbf{w}(\mathbf{u}_m) \to \mathbf{w}_0 \in \mathbb{E}_n$ since $\mathbf{u}_0 \in \mathbb{S}_+\backslash\mathbb{S}_{++}$, where $\mathbb{E}_n$ in is defined in (5.4.33).

Since $\mathbf{u}_m \in (0,\infty)^n_{\leq}$, we have $\Delta_n(\mathbf{w}_m) * \Sigma = (\mathbf{u}_m \diamond \Sigma)\,\mathrm{diag}(1/\mathbf{u}_m)$, where $\Delta_n(\mathbf{w}_m)$ is defined by (5.4.31). Thus, using (5.4.35), we obtain $\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\boldsymbol{\mu}, \mathbf{u}_m) = \boldsymbol{\mu}(\Delta_n(\mathbf{w}_m) * \Sigma)^{-1}\boldsymbol{\mu}'$. Now note that $\mathbf{w} \mapsto \Delta_n(\mathbf{w}) * \Sigma$ is a continuous mapping from $\mathbb{R}^{n-1}$ to $\mathbb{R}^{n\times n}$. Since taking the inverse of matrices is a continuous operation, by applying Lemma 5.4.10, we have

$$\mathfrak{E}_{\boldsymbol{\mu},\Sigma}(\boldsymbol{\mu}, \mathbf{u}_m) = \boldsymbol{\mu}(\Delta_n(\mathbf{w}_m) * \Sigma)^{-1}\boldsymbol{\mu}' \to \boldsymbol{\mu}(\Delta_n(\mathbf{w}_0) * \Sigma)^{-1}\boldsymbol{\mu}', \quad m \to \infty,$$

so that

$$\underline{\mathfrak{E}} = \boldsymbol{\mu}(\Delta_n(\mathbf{w}_0) * \Sigma)^{-1}\boldsymbol{\mu}' = \|\boldsymbol{\mu}(\Delta_n(\mathbf{w}_0) * \Sigma)^{-1}\|^2_{\mathrm{sym}(\Delta_n(\mathbf{w}_0)*\Sigma)},$$

where the last equality follows by using the same argument as in (5.4.36). Finally, note that $2\,\mathrm{sym}(\Delta_n(\mathbf{w}_0) * \Sigma) = \Upsilon_n(\mathbf{w}_0) * \Sigma$. By Lemma 5.4.9, $\Upsilon_n(\mathbf{w}_0)$ is nonnegative definite since $\mathbf{w}_0 \in [0,1]^{n-1}$. Thus, using the same Oppenheim's inequality argument as above, $\mathrm{sym}(\Delta_n(\mathbf{w}_0) * \Sigma)$ is positive definite, implying $\underline{\mathfrak{E}} > 0$. This completes the proof. □

5.5 Applications of Self-Decomposability Conditions

In this section, we apply the conditions for self-decomposability in Theorem 5.2.2 and non-self-decomposability in Theorem 5.3.3 to various classes of VGG^n processes.

We begin with $VGG^{n,1}$ processes.

Corollary 5.5.1. *Assume that* $\mathbf{Y} \sim VGG^{n,1}(d, \boldsymbol{\mu}, \Sigma, \mathcal{U})$.

(i) If $n = 1$*, or* $n \geq 2$ *and* $\boldsymbol{\mu} = \mathbf{0}$*, then* $\mathbf{Y}$ *is SD.*

(ii) If $n \geq 2$, $|\Sigma| \neq 0$, $\boldsymbol{\mu} \neq \mathbf{0}$ and $0 < \int_{(0,\infty)} (1 + u^{1/2}) \mathcal{U}(\mathrm{d}u) < \infty$, then $\mathbf{Y}$ is not SD.

Proof. By Example 3.2.4, $\mathbf{Y} \sim VGG^n(d\mathbf{e}, \boldsymbol{\mu}, \Sigma, \int_{(0,\infty)} \boldsymbol{\delta}_{u\mathbf{e}} \mathcal{U}(\mathrm{d}u))$. So Part (i) follows from Theorem 5.2.2. Part (ii) follows by applying Theorem 5.3.3 (iii). □

This gives a refinement of Lemma 5.1.5. Note that if the moment condition in Corollary 5.5.1, $0 < \int_{(0,\infty)} (1 + u^{1/2}) \mathcal{U}(\mathrm{d}u) < \infty$, is satisfied, then the moment condition of Lemma 5.1.5 (ii), $0 < \int_{(0,\infty)} (1 + u)^2 \mathcal{U}(\mathrm{d}u) < \infty$ when $n = 2$, or $0 < \int_{(0,\infty)} (1 + u) \mathcal{U}(\mathrm{d}u) < \infty$ when $n \geq 3$, is also satisfied. So we have improved the non-self-decomposability conditions in Proposition 3 of [Gri07b].

For VG processes, we get the following simple conditions for self-decomposability.

Corollary 5.5.2. *Assume that $\mathbf{V} \sim VG^n(b, \boldsymbol{\mu}, \Sigma)$.*

(i) If $n = 1$, or $n \geq 2$ and $\boldsymbol{\mu} = \mathbf{0}$, then $\mathbf{V}$ is SD.

(ii) If $n \geq 2$, $|\Sigma| \neq 0$, $\boldsymbol{\mu} \neq \mathbf{0}$, then $\mathbf{V}$ is not SD.

Proof. Since $\mathbf{V} \sim VGG^{n,1}(0, \boldsymbol{\mu}, \Sigma, b\boldsymbol{\delta}_b)$ by Example 3.2.4, this follows from Corollary 5.5.1, with the integral condition being trivially satisfied. □

For $VGG^{n,n}$ processes, we obtain the following self-decomposability conditions.

Corollary 5.5.3. *Assume that $\mathbf{Y} \sim VGG^{n,n}(\mathbf{d}, \boldsymbol{\mu}, \Sigma, \mathcal{U})$.*

(i) If $n = 1$, or $n \geq 2$ and $\boldsymbol{\mu} = \mathbf{0}$, then $\mathbf{Y}$ is SD.

(ii) If $n \geq 2$, $|\Sigma| \neq 0$, $\boldsymbol{\mu} \neq \mathbf{0}$, $\mathcal{U}((0,\infty)^n) > 0$ and (5.3.18) holds, then $\mathbf{Y}$ is not SD.

Proof. By Example 3.2.4, $\mathbf{Y} \sim VGG^n(\mathbf{d}, \boldsymbol{\mu}, \Sigma, \mathcal{U})$. So Parts (i) and (ii) follow from Theorem 5.2.2 and Theorem 5.3.3 (ii), respectively. □

Next, we apply the self-decomposability conditions to $WVAG$ processes. Note that these are only defined for $n \geq 2$.

Corollary 5.5.4. *Assume that $\mathbf{Y} \sim WVAG^n(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma)$.*

(i) If $\boldsymbol{\mu} = \mathbf{0}$, then $\mathbf{Y}$ is SD.

(ii) If $|\Sigma| \neq 0$ and $\boldsymbol{\mu} \neq \mathbf{0}$, then $\mathbf{Y}$ is not SD.

Proof. By Proposition 4.2.2, $\mathbf{Y} \sim VGG^n(\mathbf{0}, \boldsymbol{\mu}, \Sigma, \mathcal{U}_{a,\boldsymbol{\alpha}})$ with $\mathcal{U}_{a,\boldsymbol{\alpha}}$ defined in (4.1.5). So Part (i) follows from Theorem 5.2.2. Since $\mathcal{U}_{a,\boldsymbol{\alpha}}((0,\infty)^n \cap A) = a\boldsymbol{\delta}_{\boldsymbol{\alpha}/\|\boldsymbol{\alpha}\|^2}(A)$ for all Borel sets $A \subseteq [0,\infty)^n_*$, Part (ii) follows by applying Theorem 5.3.3 (iii). □

Example 5.5.5. It has been argued that log returns should be SD [Bin06, BK02, CGMY07]. Returning to the $WVAG$ model fitted to the bivariate S&P500 and FTSE100 data set in Section 4.6.6, we can apply the self-decomposability condition of Corollary 5.5.4 to test this claim. The log return process $\mathbf{R} \stackrel{D}{=} I\boldsymbol{\eta} + \mathbf{Y}$ is given in (4.6.1). With the notation specified there, note that $\mathbf{R}$ is SD if and only if $\mathbf{Y} \sim WVAG^n(a, \boldsymbol{\alpha}, \boldsymbol{\mu}, \Sigma)$ is SD due to the last statement in Lemma 5.1.2. This means that despite the addition of the drift term $I\boldsymbol{\eta}$, Corollary 5.5.4 is still applicable to $\mathbf{R}$.

Using ML, we obtain the parameter estimate $\boldsymbol{\mu} = (-0.0004, -0.0008)$ from Table 4.4, which is very close to $\mathbf{0}$, suggesting that $\mathbf{R}$ is likely SD. Assuming that the log returns satisfies the $WVAG$ model, a likelihood ratio test can be used to test the hypothesis $H_0 : \boldsymbol{\mu} = \mathbf{0}$ versus $H_1 : \boldsymbol{\mu} \neq \mathbf{0}$. The test statistic $D = 4.11$ is asymptotically χ^2 distributed with 2 degrees of freedom. The p-value is 0.128, so at a 5% significance level we cannot reject that the log return process $\mathbf{R}$ is SD.

We now apply these self-decomposability conditions to other VGG^n processes that have not been previously discussed.

$V\mathcal{M}\Gamma^n$ processes. For $n \geq 2$, $V\mathcal{M}\Gamma^n$ processes are defined in Section 2.5 of [BKMS17]. A Lévy process $\mathbf{Y} \stackrel{D}{=} \mathbf{B} \circ \mathbf{T}$ is a $V\mathcal{M}\Gamma^n$ *process* if $\mathbf{B} \sim BM^n(\boldsymbol{\mu}, \Sigma)$ and $\mathbf{T} \stackrel{D}{=} (G_1, \ldots, G_m)A$, where $\mathbf{B}$ and $G_k \sim \Gamma_S(b_k)$, $1 \leq k \leq m$, $m \geq 1$, are independent, and $A \in \mathbb{R}^{m \times n}$ has no zero rows and all its elements are nonnegative.

Example 5.5.6. Weakly subordinated $V\mathcal{M}\Gamma^n$ processes, which have the form $\mathbf{Y} \stackrel{D}{=} \mathbf{B} \odot \mathbf{T}$, are SD when $\boldsymbol{\mu} = \mathbf{0}$, and not SD when $|\Sigma| \neq 0$ and $\boldsymbol{\mu} \neq \mathbf{0}$. We can see this immediately in the same way as Corollary 5.5.4 because the subordinator $\mathbf{T}$ has a finitely supported Thorin measure.

VGG^n processes from beta distributions of the second kind. Let $a, b > 0$. A random variable $V(a, b) \stackrel{D}{=} G_1/G_2$, where $G_1 \sim \Gamma(a, 1)$ and $G_2 \sim \Gamma(b, 1)$ are independent, is a *beta random variable of the second kind.* Consider the univariate subordinator $T \sim GGC^1_S(0, \mathcal{U}_{a,b})$, where $\mathcal{U}_{a,b}$ is the probability measure of $V(a, b)$, so that

$$\mathcal{U}_{a,b}(\mathrm{d}u) = f_{a,b}(u)\mathrm{d}u, \quad f_{a,b}(u) := C_{a,b}u^{a-1}(1+u)^{-a-b},$$

where $C_{a,b}$ is a normalising constant (see Equation (2.2.5) [Bon92]). Note that $(1 - \ln(u))f_{a,b}(u) \sim C_{a,b}(1 - \ln(u))u^{a-1}$ as $u \downarrow 0$, which is integrable for all $a > 0$, and $u^{-1}f_{a,b}(u) \sim C_{a,b}u^{-b-2}$ as $x \to \infty$, which is integrable for all $b > 0$, so $\mathcal{U}_{a,b}$ is a Thorin measure.

Let $T_k \sim GGC^1_S(0, \mathcal{U}_{a_k,b_k})$, $a_k > 0$, $b_k > 0$, $1 \leq k \leq m$, $m \geq 1$, be independent.

Consider a VGG^n process of the form $\mathbf{Y} \stackrel{D}{=} \mathbf{B} \odot \mathbf{T}$, where $\mathbf{B} \sim BM^n(\boldsymbol{\mu}, \Sigma)$, $\mathbf{T} \stackrel{D}{=} \sum_{k=1}^m T_k \boldsymbol{\alpha}_k$ and $\boldsymbol{\alpha}_k \in [0,\infty)^n_*$, $1 \le k \le m$.

Example 5.5.7. We impose the additional assumption that $b > 1/2$. Note that $(1+u^{1/2})f_{a,b}(u)$ is eventually bounded by $(1-\ln(u))f_{a,b}(u)$ as $u \downarrow 0$, and the latter is integrable as $\mathcal{U}_{a,b}$ is a Thorin measure. Also, $(1+u^{1/2})f_{a,b}(u) \sim C_{a,b}u^{-b-1/2}$ as $u \to \infty$, which is integrable for all $b > 1/2$. Thus, we have $\int_0^\infty (1+u^{1/2})\,\mathcal{U}_{a,b}(\mathrm{d}u) < \infty$. By Theorem 5.2.2, $\mathbf{Y}$ is SD when $n = 1$, or $n \ge 2$ and $\boldsymbol{\mu} = \mathbf{0}$. By Theorem 5.3.3 (iii), $\mathbf{Y}$ is not SD when $n \ge 2$, $|\Sigma| \ne 0$, $b_k > 1/2$, $1 \le k \le m$ and $\boldsymbol{\mu} \ne \mathbf{0}$. The latter is an improvement on the non-self-decomposability condition in Lemma 5.1.5, which requires $m = 1$, $\boldsymbol{\alpha}_1 = \mathbf{e}$ and $b_1 > 1$.

CGMY processes. Let $c, g, m > 0$ and $y \in (0,2)$. A univariate Lévy process $Y \sim CGMY(c,g,m,y)$ is a *CGMY process* if it has characteristic exponent

$$\Psi_Y(\theta) = c\Gamma(-y)((m-\mathrm{i}\theta)^y - m^y + (g-\mathrm{i}\theta)^y - g^y), \quad \theta \in \mathbb{R}.$$

This process was introduced in [CGMY02] and it is a subordinated Brownian motion of the form $Y \stackrel{D}{=} B \circ T$, where $B \sim BM^1((g-m)/2, 1)$ and the $CGMY$ subordinator T are independent (see Section 3 in [MY08]). We describe T below.

We assume that $c = 1$. It has been shown in Example 8.2 of [JZ11] that the associated $CCMY$ subordinator is a Thorin subordinator, and hence $CGMY$ processes are VGG^1 processes. Here, we specify its Thorin measure. Introduce $a := 2^{y/2}/\Gamma(y)$, $a_1 := 2mg$, $a_2 := (g+m)^2/8$, where Γ is the gamma function. Let $\mathcal{U}_1(\mathrm{d}u) := au^{y/2-1}\,\mathrm{d}u$ and $\mathcal{U}_2$ be the probability measure of $a_1 + a_2V(y/2, 1/2)$, where $V(y/2,1/2)$ is a beta random variable of the second kind. Combining Equation (8.2) in [JZ11] and Theorem 3.1.1 in [Bon92], the former giving the Lévy density of T, the latter giving a formula connecting it to its Thorin measure $\mathcal{U}$, we have

$$\begin{aligned}\int_{(0,\infty)} e^{-tu}\,\mathcal{U}(\mathrm{d}u) &= a\Gamma\left(\frac{y}{2}\right)t^{-y/2}\mathbb{E}[\exp(-t(a_1 + a_2V(y/2,1/2)))] \\ &= \int_{(0,\infty)} e^{-tu}\,\mathcal{U}_1(\mathrm{d}u)\int_{(0,\infty)} e^{-tu}\,\mathcal{U}_2(\mathrm{d}u), \quad t \ge 0.\end{aligned}$$

Recognising that this is the Laplace transform of a convolution of measures, the associated $CGMY$ subordinator is $T \sim GGC^1_S(0, \mathcal{U}_1 \star \mathcal{U}_2)$, where $\star$ denotes the convolution of measures.

For $n \ge 2$, consider a multivariate $CGMY$ process $\mathbf{Y} = (Y_1,\ldots,Y_n) \stackrel{D}{=} \mathbf{B} \circ (T\mathbf{e})$, where $\mathbf{B} \sim BM(\boldsymbol{\mu}, \Sigma)$ is independent of T, $\boldsymbol{\mu} = ((g-m)/2)\mathbf{e}$, $\operatorname{diag}(\Sigma) = \mathbf{e}$, so that $Y_k \sim CGMY(1,g,m,y)$.

Example 5.5.8. Since $\int_0^\infty (1+(u+v)^{1/2})u^{y/2-1}\,\mathrm{d}u = \infty$ for all $v > 0$ and $y \in (0,2)$, the integral condition in Corollary 5.5.1 (ii) is not satisfied. The same is true of the integral condition in Lemma 5.1.5 (ii). Thus, we are unable to conclude whether or not $\mathbf{Y}$ is self-decomposable for $\boldsymbol{\mu} \neq \mathbf{0}$, and similarly for the multivariate $CGMY$ process outlined in Section 3.4 of [LS10].

Generalised hyperbolic processes. Let $(\alpha,\beta,\gamma) \in \mathbb{R}\times(0,\infty)^2 \cup (0,\infty)^2\times\{0\} \cup (-\infty,0)\times\{0\}\times(0,\infty)$. A univariate subordinator $T \sim GIG_S(\alpha,\beta,\gamma)$ is a *generalised inverse Gaussian subordinator* if $T \sim GGC_S^1(0,\mathcal{U}_{\alpha,\beta,\gamma})$, where

$$\begin{aligned}\mathcal{U}_{\alpha,\beta,\gamma}(\mathrm{d}u) &:= \alpha^+\boldsymbol{\delta}_\beta(\mathrm{d}u) + 2\gamma\mathbf{1}_{(\beta,\infty)}(u)g_{|\alpha|}(4\gamma(u-\beta))\,\mathrm{d}u,\\ g_\rho(u) &:= 2(\pi^2 u(J_\rho^2(u^{1/2}) + Y_\rho^2(u^{1/2})))^{-1}, \quad \rho \geq 0,\end{aligned} \tag{5.5.1}$$

and J_ρ and Y_ρ are Bessel functions of the first and second kind, respectively. GIG subordinators were originally characterised as Thorin subordinators in [Hal79] though the representation above is taken from Example 1 in [Gri07b] and Remark 2.8 in [BKMS17]. If $\gamma = 0$, then the GIG subordinator reduces to a gamma subordinator $T \sim \Gamma_S(\alpha,\beta)$. A Lévy process $\mathbf{Y} \sim GH^n(\alpha,\beta,\gamma,\boldsymbol{\mu},\Sigma)$ is a *generalised hyperbolic process* if $\mathbf{Y} \overset{D}{=} \mathbf{B}\circ(T\mathbf{e})$, where $\mathbf{B} \sim BM^n(\boldsymbol{\mu},\Sigma)$ and $T \sim GIG_S(\alpha,\beta,\gamma)$ are independent.

Example 5.5.9. Assume that $n \geq 2$ and $|\Sigma| \neq 0$. As noted in Example 1 of [Gri07b], $g_\rho(u) \sim cu^{-1/2}$, $\rho \geq 0$, as $u \to \infty$ for some constant $c > 0$, so the integral condition in Corollary 5.5.1 (ii), although less stringent than the condition in Lemma 5.1.5 (ii), is not satisfied unless $\gamma = 0$. Thus, as in [Gri07b], no conclusion about the self-decomposability of $\mathbf{Y}$ can be drawn when $\gamma \neq 0$, $\boldsymbol{\mu} \neq \mathbf{0}$.

However, we can numerically examine the function $\mathfrak{H}_\mathbf{s}(r)$ defined in (5.3.14) for particular parameter values. For $\mathbf{s} \in \mathbb{S}^*$, we have

$$\begin{aligned}\mathfrak{H}_\mathbf{s}(r) = \frac{2}{(2\pi)^{n/2}}\frac{\exp(r\langle\mathbf{s},\boldsymbol{\mu}\rangle_{\Sigma^{-1}})}{\|\mathbf{s}\|_{\Sigma^{-1}}^n|\Sigma|^{1/2}}\bigg(&\alpha^+\mathfrak{K}_{n/2}(r(2n\beta+\|\boldsymbol{\mu}\|_{\Sigma^{-1}})^{1/2}\|\mathbf{s}\|_{\Sigma^{-1}})\\ &+ 2\gamma\int_{(\beta,\infty)}\mathfrak{K}_{n/2}(r(2nu+\|\boldsymbol{\mu}\|_{\Sigma^{-1}})^{1/2}\|\mathbf{s}\|_{\Sigma^{-1}})g_{|\alpha|}(4\gamma(u-\beta))\,\mathrm{d}u\bigg),\end{aligned}$$

where $\mathfrak{K}_{n/2}$ is defined in (1.3.2) and $g_{|\alpha|}$ is defined in (5.5.1). Recall that due to Lemma 5.3.1, the proof of Theorem 5.3.3 proceeds by showing that $r \mapsto \mathfrak{H}_\mathbf{s}(r)$ is increasing at the origin on a Borel set $\mathbb{B} \subseteq \mathbb{S}^*$ of positive Lebesgue surface measure. Suppose that $\mathbf{Y} \sim GH^3(-1,2,0.5,(-5,0,0),\mathrm{diag}(0.05,1,1))$. A plot of $r \mapsto \mathfrak{H}_\mathbf{s}(r)$ at $\mathbf{s} = (-1,0,0)$ is given in Figure 5.1. This numerical experiment suggests that $\mathfrak{H}_\mathbf{s}$ may be decreasing at the origin but strictly increasing at an alternative point. If this behaviour extends to a set $\mathbf{s} \in \mathbb{S}^*$ of strictly positive Lebesgue surface measure, then $\mathbf{Y}$ cannot be self-decomposable.

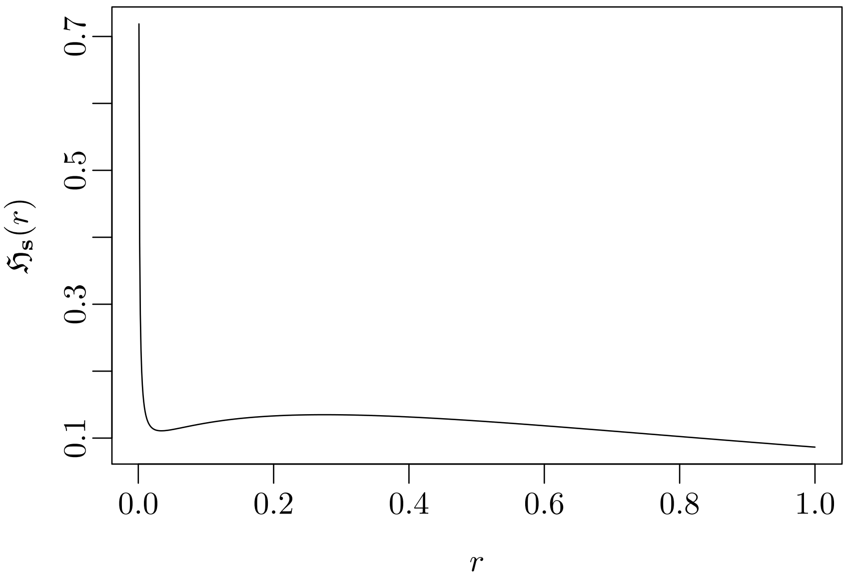

Figure 5.1: Plot of $r \mapsto \mathfrak{H}_{\mathbf{s}}(r)$ for $GH^3(-1, 2, 0.5, (-5, 0, 0), \mathrm{diag}(0.05, 1, 1))$ at $\mathbf{s} = (-1, 0, 0)$.

As these examples demonstrate, the non-self-decomposability conditions in Theorem 5.3.3 can be easily applied to $WVAG$ processes and processes with finitely supported Thorin measure. However, they fail to be applicable for many high activity VGG^n processes, despite improving on the conditions in Lemma 5.1.5 (ii).

In addition, Example 5.5.9 and Figure 5.1 suggests that proving or disproving the conjecture that $\boldsymbol{\mu} = \mathbf{0}$ is equivalent to self-decomposability for VGG^n processes will likely require methods that consider the function $r \mapsto \mathfrak{H}_{\mathbf{s}}(r)$ on its entire domain, not just near the origin, and that relax the moment conditions on the underlying Thorin measure.

Concluding Remarks

While it is widely known that strong subordination produces Lévy processes when the subordinate has independent components or the subordinator has indistinguishable components, we have proved that these conditions are necessary in a wide range of cases. Our main contribution is the introduction of weak subordination, an operation that always creates a Lévy process, while extending strong subordination by matching it in law in the aforementioned cases. Numerous properties of weak subordination have been derived.

We have used weak subordination to generalise the class of $VGG^{n,1}$ and $VGG^{n,n}$ processes with the superclass of VGG^n processes, and characterised the laws of the latter. In particular, we focused on $WVAG$ processes, and we studied Fourier invertibility and calibration methods for these. Based on moment formulas and a fit of the models to an S&P500-FTSE100 data set, we found that $WVAG$ processes exhibit a wider range of dependence and produces a significantly better fit than VAG processes constructed by strong subordination.

In addition, we have proved sufficient conditions as well as necessary conditions for the self-decomposability of VGG^n processes, extending and improving the work of Grigelionis [Gri07b], who obtained analogous results for $VGG^{n,1}$ processes.

Outside of our applications, weak subordination has been used in financial modelling. In Michaelsen and Szimayer [MS18], various marginal consistent dependence models have been constructed by weak subordination. In Madan [Mad18], log returns modelled using $WVAG$ processes were applied in instantaneous portfolio theory.

There are a variety of research directions for weak subordination. On the theoretical side, there are several open questions regarding the connection between strong and weak subordination. Weakening the assumptions of Proposition 1.3.6 and 2.3.29 is of interest to more completely determine the necessary and sufficient conditions for strong subordination to create a Lévy process, and when it is possible for a weakly subordinated process, or any Lévy process, to have time marginals that match those of a strongly subordinated process. It is also not known if weak and strong subordination coincide in law in all the cases where strong subordination produces a Lévy process, though in light of the construction in Section 2.1, we conjecture that this is true.

Furthermore, whether or not the self-decomposability of a VGG^n process is equivalent to its Brownian motion subordinate being driftless remains open. Related to this, it would be useful to relax the moment conditions on the Thorin measure in the necessary conditions of Theorem 5.3.3. Another future direction could be to extend these results about VGG^n processes to operator self-decomposability and to find conditions for their inclusion in Urbanik's L classes. In the context of multivariate subordination, sufficient conditions for this were derived in [BNPS01]. Studying the consistency and asymptotics of the estimators for $WVAG$ processes is another potential area of research.

On the practical side, additional weakly subordinated models, such as extending the multivariate GH and $CGMY$ processes, could be explored. Other possibilities include using subordinates other than Brownian motion, such as stable processes, or subordinators that do not arise via ray subordination. The latter has been considered in Michaelsen [Mic18], where subordinators with arbitrary marginal components and dependence specified by a Lévy copula were studied.

Given that the assumption of iid log returns required for the $WVAG$ model in Section 4.6 may be unrealistic in practice, we could alternatively consider related models with autocorrelation for future research. Recall that in Example 5.5.5, we fail to reject that the log returns of the S&P500-FTSE100 dataset are self-decomposable. This suggests models using the one-to-one correspondence between self-decomposable distributions and Lévy-driven Ornstein-Uhlenbeck processes may be appropriate. To be precise, log returns could be modelled using a multivariate Lévy-driven Ornstein-Uhlenbeck process with a self-decomposable $WVAG$ stationary distribution and having exponentially decaying autocorrelation. However, more research following on from [BNS01] would be needed regarding parameter estimation for these processes, which is complicated by the fact that the log returns are no longer independent.

Appendix A

Miscellaneous Results

A.1 Modified Bessel Functions of the Second Kind

This section revises some properties of the modified Bessel function K_ρ of the second kind of order $\rho \geq 0$. We define

$$\mathfrak{K}_\rho(r) := r^\rho K_\rho(r) = 2^{\rho-1} \int_0^\infty x^{\rho-1} \exp\left(-x - \frac{r^2}{4x}\right) \mathrm{d}x, \quad \rho \geq 0, \quad r > 0. \tag{A.1.1}$$

(see Equation (3.471)–9 in [GR15]). In particular, $K_{1/2}(r) = (\pi/2)^{1/2} e^{-r} r^{-1/2}$, $r > 0$, (see Equation (8.469)–3 in [GR15]) so that

$$\mathfrak{K}_{1/2}(r) = \left(\frac{\pi}{2}\right)^{1/2} e^{-r}, \quad r > 0. \tag{A.1.2}$$

We use the following facts regarding Bessel functions.

Lemma A.1.1.

(i) For $\rho \geq 0$, $r \mapsto \mathfrak{K}_\rho(r)$ is nonnegative and nonincreasing.

(ii) We have, $K_0(r) \sim -\ln(r)$ as $r \downarrow 0$.

(iii) For $\rho > 0$, $\mathfrak{K}_\rho$ is uniformly bounded by $\mathfrak{K}_\rho(0+) = 2^{\rho-1}\Gamma(\rho)$.

(iv) For $\rho \geq 0$, $K_\rho(r) \sim K_{1/2}(r) = (\pi/2)^{1/2} e^{-r} r^{-1/2}$ as $r \to \infty$.

(v) For $\rho \geq 0$, $a, \theta > 0$ and any n-dimensional Thorin measure $\mathcal{U}$, we have

$$\int_{(0,\infty)^n} \mathfrak{K}_\rho(a\|\mathbf{u}\|^\theta)\, \mathcal{U}(\mathrm{d}\mathbf{u}) < \infty. \tag{A.1.3}$$

(vi) For $\rho \geq 1$, $(\mathrm{d}/\mathrm{d}r)\mathfrak{K}_\rho(r) = -r\mathfrak{K}_{\rho-1}(r)$, $r > 0$.

(vii) For $\rho = (n-2)/2$, $n \geq 2$, we have

$$0 \leq \mathfrak{K}_{n/2}(r) - \mathfrak{K}_{n/2}(2r) \leq \frac{3}{2} r^2 \mathfrak{K}_\rho(r), \quad r > 0. \tag{A.1.4}$$

(viii) For $\rho \geq 0$, $\lim_{r \downarrow 0} r\mathfrak{K}_\rho(r) = 0$.

(ix) For $\rho \geq 0$, $\sup_{r>0} r\mathfrak{K}_\rho(r) < \infty$.

Proof. (i). This follows immediately from (A.1.1).

(ii)–(iii). See Equation (A.3) in [Gau14]. For an alternative proof of Part (iii), note that (A.1.1) implies $\mathfrak{K}_\rho(r) = 2^{\rho-1}\Gamma(\rho)\mathbb{E}[\exp(-r^2/(4G))]$, $r > 0$, where $G \sim \Gamma(\rho, 1)$ and Γ is the gamma function (see Equation (1.l) in [PY81]).

(iv). See Equation (8.451)–6 in [GR15].

(v). By Parts (ii) and (iii), $\mathfrak{K}_\rho(ar^\theta)$ is eventually bounded by $c(1 - \ln(r))$ as $r \downarrow 0$ for some constant $c > 0$. By Part (iv), $\mathfrak{K}_\rho(ar^\theta)$ is eventually bounded by $1/r$ as $r \to \infty$. So noting (3.1.1), the integral is bounded.

(vi). See Equation (A.13) in [Gau14].

(vii). Let $r > 0$. The first inequality follows from $s \mapsto \mathfrak{K}_\rho(s)$ being nonincreasing. For the second inequality, note that $(\mathrm{d}/\mathrm{d}r)\mathfrak{K}_{n/2}(r) = -r\mathfrak{K}_\rho(r)$ by Part (vi). Now using the fundamental theorem of calculus, followed by the nonincreasingness of $s \mapsto \mathfrak{K}_\rho(s)$, we get

$$\mathfrak{K}_{n/2}(r) - \mathfrak{K}_{n/2}(2r) = \int_r^{2r} s\mathfrak{K}_\rho(s)\,\mathrm{d}s \leq \mathfrak{K}_\rho(r) \int_r^{2r} s\,\mathrm{d}s = \frac{3}{2} r^2 \mathfrak{K}_\rho(r).$$

(viii). For $\rho = 0$, this follows from Part (ii). For $\rho > 0$, this follows from Part (iii).

(ix). By Part (viii), $\lim_{r\downarrow 0} r\mathfrak{K}_\rho(r) = 0$. By Part (iv), $\lim_{r\to\infty} r\mathfrak{K}_\rho(r) = 0$. Since $r \mapsto r\mathfrak{K}_\rho(r)$, $r > 0$, is a continuous function, it must have a finite supremum. □

A.2 Linear Algebra

Recall that $*$ denotes the Hadamard product of matrices. The following two inequalities are known as Oppenheim's and Hadamard's inequalities, respectively, and are critical for proving conditions for the non-self-decomposability of VGG^n processes in Sections 5.3 and 5.4.

Lemma A.2.1. *For covariance matrices $A = (A_{kl}) \in \mathbb{R}^{n\times n}$ and $B = (B_{kl}) \in \mathbb{R}^{n\times n}$, we have*

$$\left(\prod_{k=1}^n A_{kk}\right)|B| \leq |A * B| \leq \prod_{k=1}^n A_{kk}B_{kk}. \tag{A.2.1}$$

Proof. See Theorems 3.6.3 and 3.7.5 in [BR97]. □

The condition that a covariance matrix $\Sigma = (\Sigma_{kl}) \in \mathbb{R}^{n\times n}$ is invertible is invoked in several places throughout the text. The next lemma gives some useful results for this situation. It implies that no component of $\mathbf{B} \sim BM^n(\boldsymbol{\mu}, \Sigma)$ can degenerate to a zero process.

Lemma A.2.2. *Let $\Sigma \in \mathbb{R}^{n\times n}$ be a covariance matrix. Then*

(i) Σ is invertible if and only if $|\Sigma| > 0$;

(ii) Σ is invertible if and only if Σ is positive definite;

(iii) If Σ is invertible then $\Sigma_{kk} > 0$ for all $1 \le k \le n$.

Proof. (i). Since Σ is a covariance matrix, $|\Sigma| \ge 0$. Since Σ is invertible, $|\Sigma| \ne 0$. Conversely, $|\Sigma| > 0$ implies that Σ is invertible.

(ii). By Part (i), we have $0 < |\Sigma| = \prod_{k=1}^n \lambda_k$, where $\lambda_1, \dots \lambda_n$ are the eigenvalues of Σ, and they must be nonnegative since Σ is nonnegative definite. For this to hold, we must have $\lambda_k > 0$ for all $1 \le k \le n$, and hence Σ is positive definite. Conversely, all positive definite matrices are invertible.

(iii). By Part (i) and Hadamard's inequality (see Lemma A.2.1), we have $0 < |\Sigma| \le \prod_{k=1}^n \Sigma_{kk}$. This inequality would be violated unless $\Sigma_{kk} > 0$ for all $1 \le k \le n$. □

For a matrix $A \in \mathbb{R}^{n\times n}$, let $\mathrm{sym}(A) := (A + A')/2 \in \mathbb{R}^{n\times n}$ denote the symmetrisation of A.

Lemma A.2.3. *If $\mathbf{x} \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, then $\|\mathbf{x}\|_A^2 = \|\mathbf{x}\|_{\mathrm{sym}(A)}^2$*

Proof. We have $A = (A - A')/2 + \mathrm{sym}(A)$, and substituting this into $\mathbf{x}A\mathbf{x}'$ gives $\mathbf{x}\,\mathrm{sym}(A)\mathbf{x}'$, which completes the proof. □

A.3 Analysis and Measure Theory

We state the transformation theorem below.

Proposition A.3.1. *Let $\mathbb{X}$ and $\mathbb{Y}$ be measure spaces, $\mathcal{X}$ be a measure on $\mathbb{X}$, $f : \mathbb{X} \to \mathbb{Y}$ and $g : \mathbb{Y} \to \mathbb{R}$ be measurable functions, where g is nonnegative, then*

$$\int_{\mathbb{Y}} g(y)\,(\mathcal{X} \circ f^{-1})(\mathrm{d}y) = \int_{\mathbb{X}} g(f(x))\,\mathcal{X}(\mathrm{d}x).$$

In particular, g is $(\mathcal{X} \circ f^{-1})$-integrable if and only if $g \circ f$ is $\mathcal{X}$-integrable.

Proof. See Theorem 19.1 and Corollary 19.2 in [Bau92]. ☐

The following statement is Fatou's lemma for random variables.

Lemma A.3.2. *For random variables $X_m \geq 0$, $m \geq 1$, we have*

$$\liminf_{m\to\infty} \mathbb{E}[X_m] \geq \mathbb{E}\Big[\liminf_{m\to\infty} X_m\Big].$$

Proof. See Theorem 9.1 (e) in [JP04]. ☐

The next lemma is the polar decomposition of the Lebesgue measure.

Lemma A.3.3. *If $f : \mathbb{R}^n \to \mathbb{R}$ is Borel measurable and Lebesgue integrable, then*

$$\int_{\mathbb{R}^n} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\mathbb{S}} \int_{(0,\infty)} r^n f(r\mathbf{s})\,\frac{\mathrm{d}r}{r}\mathrm{d}\mathbf{s}.$$

Proof. See Equation (4) in [Gri07b]. ☐

Lastly, we have an elementary and useful complex number inequality.

Lemma A.3.4. *If $z \in \mathbb{C}$ and $\Re z \leq 0$, then $|e^z - 1| \leq |z|$.*

Proof. Using $1 - e^{-x} \leq x$, $1 - \cos x \leq x^2/2$, $x \in \mathbb{R}$, and assuming $\Re z \leq 0$, we have

$$|e^z - 1|^2 = (1 - e^{\Re z})^2 + 2e^{\Re z}(1 - \cos(\Im z)) \leq (\Re z)^2 + (\Im z)^2 = |z|^2,$$

which implies the result. ☐

Appendix B

Calibration Code

This appendix reproduces some illustrative portions of the code that was used to implement the calibration methods outlined in Section 4.6. This includes the simulation of $WVAG$ processes and the estimation of their parameters using ML, DME and MOM. We also include the computation of the KS and χ^2 statistic, while the computation of the $-\ln L$ statistic is not included as it simply involves evaluating the objective function in the ML code.

The code is written in the programming language R.

Functions

```
#Libraries
library(VarianceGamma)
library(pracma)
library(MASS)
library(Peacock.test)

#Functions
sim.vag <- function(a,alpha1,alpha2,mu1,mu2,S11,S22,rho,t.max=1000){
    S12 <- rho*sqrt(S11*S22)
    if(a >= min(1/c(alpha1,alpha2))){
        return(NA)
    }
    adjustedmu <- c(mu1,mu2)
    Sigma <- array(c(S11,S12,S12,S22),c(2,2))
    adjustedmu.list <- Sigma.list <- list()
    #V0
    S0 <- rgamma(n=t.max,shape=a,rate=a)
    adjustedmu0 <- a*c(alpha1,alpha2)*adjustedmu
    Sigma0 <- a*outer(c(alpha1,alpha2),c(alpha1,alpha2),pmin)*Sigma
    N0 <- t(mvrnorm(n=t.max,mu=c(0,0),Sigma=Sigma0))
    V0 <- rbind(sqrt(S0)*N0[1,]+S0*adjustedmu0[1],sqrt(S0)*N0[2,]+S0*adjustedmu0[2])
    #V1
    S1 <- rgamma(n=t.max,shape=(1-a*alpha1)/alpha1,rate=(1-a*alpha1)/alpha1)
    adjustedmu1 <- S1*(1-a*alpha1)*mu1
    Sigma1 <- S1*(1-a*alpha1)*S11
```

```
    V1 <- rnorm(n=t.max,mean=adjustedmu1,sd=sqrt(Sigma1))
    #V2
    S2 <- rgamma(n=t.max,shape=(1-a*alpha2)/alpha2,rate=(1-a*alpha2)/alpha2)
    adjustedmu2 <- S2*(1-a*alpha2)*mu2
    Sigma2 <- S2*(1-a*alpha2)*S22
    V2 <- rnorm(n=t.max,mean=adjustedmu2,sd=sqrt(Sigma2))
    returns <- t(V0)+cbind(V1,V2)
    colnames(returns) <- c("R1","R2")
    return(returns)
}
pvg.2 <- function(y,theta,sigma,nu,vgC){
    const <- 2/(sigma*sqrt(2*pi)*nu^(1/nu)*gamma(1/nu))*(1/sqrt(2*sigma^2/nu+theta^2))^(1/nu-1/2)
    vg.f <- function(x){
        if(x==0){
            x <- 1e-8
            rv1 <- const*exp(theta*(x-vgC)/sigma^2)*abs(x-vgC)^(1/nu-1/2)*
            besselK((abs(x-vgC)*sqrt(2*sigma^2/nu+theta^2))/sigma^2 ,1/nu-1/2)
            x <- -1e-8
            rv2 <- const*exp(theta*(x-vgC)/sigma^2)*abs(x-vgC)^(1/nu-1/2)*
            besselK((abs(x-vgC)*sqrt(2*sigma^2/nu+theta^2))/sigma^2 ,1/nu-1/2)
            rv <- (rv1+rv2)/2
        }else{
            rv <- const*exp(theta*(x-vgC)/sigma^2)*abs(x-vgC)^(1/nu-1/2)*
            besselK((abs(x-vgC)*sqrt(2*sigma^2/nu+theta^2))/sigma^2 ,1/nu-1/2)
        }
        return(rv)
    }
    integrate(Vectorize(vg.f),lower=-Inf,upper=y)$val
}
obj.fn <- function(x,q,k){
    e.x2 <- exp(x[2])
    e.x3 <- exp(x[3])
    f <- numeric(n.qu)
    for(j in 1:n.qu){
        tmp <- try(pvg(k[j],theta=x[1],sigma=e.x2,nu=e.x3,vgC=x[4])-q[j])
        if(is.numeric(tmp)==TRUE){
            f[j] <- tmp
        }else{
            tmp2 <- try(pvg.2(k[j],theta=x[1],sigma=e.x2,nu=e.x3,vgC=x[4])-q[j])
            if(is.numeric(tmp2)==TRUE){
                f[j] <- tmp2
            }else{
                return(Inf)
            }
        }
    }
    wt <- 1
    return(sum(wt*f^2))
}
obj.fn.2d <- function(a.est,rho.est,alpha1.est,alpha2.est,mu1.est,mu2.est,
S11.est,S22.est,eta1.est,eta2.est,quan.q,k1,k2){
    vag <- sim.vag(a=a.est,rho=rho.est,
    alpha1=alpha1.est,alpha2=alpha2.est,mu1=mu1.est,mu2=mu2.est,S11=S11.est,
    S22=S22.est,t.max=n.est)
    if(sum(is.na(vag))>0){
        return(NA)
    }
```

```
    R1.est <- vag[,1]+eta1.est
    R2.est <- vag[,2]+eta2.est
    quan.q.est <- array(NA,c(length(k1),length(k2)))
    for(i in 1:length(k1)){
        for(j in 1:length(k2)){
            k.tmp <- trans.mat%*%c(k1[i],k2[j])
            quan.q.est[i,j] <- sum(R1.est<=k.tmp[1,] & R2.est<=k.tmp[2,])/n.est
        }
    }
    wt <- 1
    error <- sum(wt*(quan.q.est-quan.q)^2)
    rm(R1.est,R2.est)
    gc()
    return(error)
}
obj.fn.2d <- Vectorize(obj.fn.2d,vectorize.args=c("a.est","rho.est"))
get.to.min <- function(x,loess.fit=l2){
    loess.df2 <- data.frame(grid1=x[1],grid2=x[2])
    loess.pred <- predict(loess.fit,newdata=loess.df2)
    return(as.numeric(loess.pred))
}
scale <- 1
delta <- 1
```

Method of Moments

```
n.sim <- 100
set.seed(46473448)
boot.par.init <- array(NA,c(n.sim,10))
boot.par <- array(NA,c(n.sim,10))
optim.list <- list()
tol.val <- 1e-4
for(i.sim in 1:n.sim){
    return.process <- sim.vag(a.true,alpha1.true,alpha2.true,mu1.true,mu2.true,S11.true,S22.true
    ,rho.true,t.max=1000)
    R1 <- return.process[,1]+eta1.true
    R2 <- return.process[,2]+eta2.true
    STARTINGVAL1 = 1
    STARTINGVAL21 = var(R1)
    STARTINGVAL22 = var(R2)
    # Marginal paramters 1
    est.m <- mean(R1)
    est.v <- var(R1)
    est.s <- mean((R1-mean(R1))^3)
    est.k <- mean((R1-mean(R1))^4)
    mom.eq <- function(x){
        alpha1 <- x[1]
        S11 <- x[2]
        mu1 <- x[3]
        eta1 <- x[4]
        component1 <- mu1+eta1 - est.m
        component2 <- S11+alpha1*mu1^2 - est.v
        component3 <- 3*alpha1*S11*mu1+2*alpha1^2*mu1^3 - est.s
        component4 <- 3*S11^2+3*alpha1*(S11^2+2*S11*mu1^2)+3*alpha1^2*(4*S11*mu1^2+mu1^4)+6*alpha1^3
        *mu1^4 - est.k
        return(c(component1,component2,component3,component4))
```

```
}
soln1 <- try(fsolve(mom.eq,c(STARTINGVAL1,STARTINGVAL21,0,0),maxiter=10000)$x)
if(is.character(soln1)==FALSE){
    alpha1 <- soln1[1]
    S11 <- soln1[2]
    mu1 <- soln1[3]
    eta1 <- soln1[4]
    tol <- sum(fsolve(mom.eq,c(STARTINGVAL1,STARTINGVAL21,0,0),maxiter=10000)$fval^2)
    if(alpha1 <= 0 | S11 <= 0 | tol > tol.val){
        mom.eq <- function(x){
            alpha1 <- exp(x[1])
            S11 <- exp(x[2])
            mu1 <- x[3]
            eta1 <- x[4]
            component1 <- mu1+eta1 - est.m
            component2 <- S11+alpha1*mu1^2 - est.v
            component3 <- 3*alpha1*S11*mu1+2*alpha1^2*mu1^3 - est.s
            component4 <- 3*S11^2+3*alpha1*(S11^2+2*S11*mu1^2)+3*alpha1^2*(4*S11*mu1^2+mu1^4)+6*
            alpha1^3*mu1^4 - est.k
            return(sum(c(component1,component2,component3,component4)^2))
        }
        soln1 <- optim(c(log(STARTINGVAL1),log(STARTINGVAL21),0,0),mom.eq,
        control=list(maxit=1e4))$par
        alpha1 <- exp(soln1[1]); S11 <- exp(soln1[2]); mu1 <- soln1[3]; eta1 <- soln1[4]
    }
}else{
    mom.eq <- function(x){
        alpha1 <- exp(x[1])
        S11 <- exp(x[2])
        mu1 <- x[3]
        eta1 <- x[4]
        component1 <- mu1+eta1 - est.m
        component2 <- S11+alpha1*mu1^2 - est.v
        component3 <- 3*alpha1*S11*mu1+2*alpha1^2*mu1^3 - est.s
        component4 <- 3*S11^2+3*alpha1*(S11^2+2*S11*mu1^2)+3*alpha1^2*(4*S11*mu1^2+mu1^4)+6*
        alpha1^3*mu1^4 - est.k
        return(sum(c(component1,component2,component3,component4)^2))
    }
    soln1 <- optim(c(log(STARTINGVAL1),log(STARTINGVAL21),0,0),
    mom.eq,control=list(maxit=1e4))$par
    alpha1 <- exp(soln1[1]); S11 <- exp(soln1[2]); mu1 <- soln1[3]; eta1 <- soln1[4]
}
# Marginal paramters 2
est.m <- mean(R2)
est.v <- var(R2)
est.s <- mean((R2-mean(R2))^3)
est.k <- mean((R2-mean(R2))^4)
mom.eq <- function(x){
    alpha2 <- x[1]
    S22 <- x[2]
    mu2 <- x[3]
    eta2 <- x[4]
    component1 <- mu2+eta2 - est.m
    component2 <- S22+alpha2*mu2^2 - est.v
    component3 <- 3*alpha2*S22*mu2+2*alpha2^2*mu2^3 - est.s
    component4 <- 3*S22^2+3*alpha2*(S22^2+2*S22*mu2^2)+3*alpha2^2*(4*S22*mu2^2+mu2^4)+6*
    alpha2^3*mu2^4 - est.k
```

```
    return(c(component1,component2,component3,component4))
}
soln2 <- try(fsolve(mom.eq,c(STARTINGVAL1,STARTINGVAL22,0,0),maxiter=10000)$x)
if(is.character(soln2)==FALSE){
    alpha2 <- soln2[1]
    S22 <- soln2[2]
    mu2 <- soln2[3]
    eta2 <- soln2[4]
    tol <- sum(fsolve(mom.eq,c(STARTINGVAL1,STARTINGVAL22,0,0),maxiter=10000)$fval^2)
    if(alpha2 <= 0 | S22 <= 0 | tol > tol.val){
        mom.eq <- function(x){
            alpha2 <- exp(x[1])
            S22 <- exp(x[2])
            mu2 <- x[3]
            eta2 <- x[4]
            component1 <- mu2+eta2 - est.m
            component2 <- S22+alpha2*mu2^2 - est.v
            component3 <- 3*alpha2*S22*mu2+2*alpha2^2*mu2^3 - est.s
            component4 <- 3*S22^2+3*alpha2*(S22^2+2*S22*mu2^2)+3*alpha2^2*(4*S22*mu2^2+mu2^4)+6*
            alpha2^3*mu2^4 - est.k
            return(sum(c(component1,component2,component3,component4)^2))
        }
        soln2 <- optim(c(log(STARTINGVAL1),log(STARTINGVAL22),0,0),mom.eq,
        control=list(maxit=1e4))$par
        alpha2 <- exp(soln2[1]); S22 <- exp(soln2[2]); mu2 <- soln2[3]; eta2 <- soln2[4]
    }
}else{
    mom.eq <- function(x){
        alpha2 <- exp(x[1])
        S22 <- exp(x[2])
        mu2 <- x[3]
        eta2 <- x[4]
        component1 <- mu2+eta2 - est.m
        component2 <- S22+alpha2*mu2^2 - est.v
        component3 <- 3*alpha2*S22*mu2+2*alpha2^2*mu2^3 - est.s
        component4 <- 3*S22^2+3*alpha2*(S22^2+2*S22*mu2^2)+3*alpha2^2*(4*S22*mu2^2+mu2^4)+6*
        alpha2^3*mu2^4 - est.k
        return(sum(c(component1,component2,component3,component4)^2))
    }
    soln2 <- optim(c(log(STARTINGVAL1),log(STARTINGVAL22),0,0),mom.eq,
    control=list(maxit=1e4))$par
    alpha2 <- exp(soln2[1]); S22 <- exp(soln2[2]); mu2 <- soln2[3]; eta2 <- soln2[4]
}
STARTINGVAL23 = cov(R1,R2)
# Joint parameters
est.cv <- cov(R1,R2)
est.ck <- mean((R1-mean(R1))^2*(R2-mean(R2))^2)
mom.eq <- function(x){
    a <- x[1]
    S12 <- x[2]
    cokurt.formula <- 2*a^2*(min(alpha1,alpha2)^2*S12^2+2*alpha1*alpha2*min(alpha1,alpha2)*
    S12*mu1*mu2+alpha1^2*alpha2^2*mu1^2*mu2^2)+
    2*a*(min(alpha1,alpha2)^2*S12^2+4*alpha1*alpha2*min(alpha1,alpha2)*S12*mu1*mu2+3*
    alpha1^2*alpha2^2*mu1^2*mu2^2+
    alpha1^2*alpha2*S22*mu1^2+alpha1*alpha2^2*S11*mu2^2+0.5*alpha1*alpha2*S11*S22)+
    alpha1*alpha2*mu1^2*mu2^2+alpha1*S22*mu1^2+alpha2*S11*mu2^2+S11*S22
    component1 <- a*(min(alpha1,alpha2)*S12+alpha1*alpha2*mu1*mu2) - est.cv
```

```
    component2 <- cokurt.formula - est.ck
    return(c(component1,component2))
}
soln3 <- try(fsolve(mom.eq,c(STARTINGVAL1,STARTINGVAL23),maxiter=10000)$x)
if(is.character(soln3)==FALSE){
    a <- soln3[1]
    S12 <- soln3[2]
    tol <- sum(fsolve(mom.eq,c(STARTINGVAL1,STARTINGVAL23),maxiter=10000)$fval^2)
    if(!(0 <a & a<min(1/alpha1,1/alpha2)) | !(abs(S12/sqrt(S11*S22))<1) | tol > tol.val){
        mom.eq <- function(x){
            a <- x[1]
            S12 <- x[2]
            if(!(0<a & a<min(1/alpha1,1/alpha2) & abs(S12/sqrt(abs(S11*S22)))<1)){
                return(Inf)
            }
            cokurt.formula <- 2*a^2*(min(alpha1,alpha2)^2*S12^2+2*alpha1*alpha2*min(alpha1,
            alpha2)*S12*mu1*mu2+alpha1^2*alpha2^2*mu1^2*mu2^2)+
            2*a*(min(alpha1,alpha2)^2*S12^2+4*alpha1*alpha2*min(alpha1,alpha2)*S12*mu1*mu2+3*
            alpha1^2*alpha2^2*mu1^2*mu2^2+
            alpha1^2*alpha2*S22*mu1^2+alpha1*alpha2^2*S11*mu2^2+0.5*alpha1*alpha2*S11*S22)+
            alpha1*alpha2*mu1^2*mu2^2+alpha1*S22*mu1^2+alpha2*S11*mu2^2+S11*S22
            component1 <- a*(min(alpha1,alpha2)*S12+alpha1*alpha2*mu1*mu2) - est.cv
            component2 <- cokurt.formula - est.ck
            return(sum(c(component1,component2)^2))
        }
        if(!(-sqrt(S11*S22) < STARTINGVAL23 & STARTINGVAL23 < sqrt(S11*S22))){
            STARTINGVAL23 <- 0
        }
        soln3 <- optim(c(min(1/alpha1,1/alpha2)/2,STARTINGVAL23),mom.eq,
        control=list(maxit=1e4))$par
        a <- soln3[1]; S12 <- soln3[2]
    }
}else{
    mom.eq <- function(x){
        a <- x[1]
        S12 <- x[2]
        if(!(0<a & a<min(1/alpha1,1/alpha2) & abs(S12/sqrt(abs(S11*S22)))<1)){
            return(Inf)
        }
        cokurt.formula <- 2*a^2*(min(alpha1,alpha2)^2*S12^2+2*alpha1*alpha2*min(alpha1,alpha2)*
        S12*mu1*mu2+alpha1^2*alpha2^2*mu1^2*mu2^2)+
        2*a*(min(alpha1,alpha2)^2*S12^2+4*alpha1*alpha2*min(alpha1,alpha2)*S12*mu1*mu2+3*
        alpha1^2*alpha2^2*mu1^2*mu2^2+
        alpha1^2*alpha2*S22*mu1^2+alpha1*alpha2^2*S11*mu2^2+0.5*alpha1*alpha2*S11*S22)+
        alpha1*alpha2*mu1^2*mu2^2+alpha1*S22*mu1^2+alpha2*S11*mu2^2+S11*S22
        component1 <- a*(min(alpha1,alpha2)*S12+alpha1*alpha2*mu1*mu2) - est.cv
        component2 <- cokurt.formula - est.ck
        return(sum(c(component1,component2)^2))
    }
    if(!(-sqrt(S11*S22) < STARTINGVAL23 & STARTINGVAL23 < sqrt(S11*S22))){
        STARTINGVAL23 <- 0
    }
    soln3 <- optim(c(min(1/alpha1,1/alpha2)/2,STARTINGVAL23),mom.eq,control=list(maxit=1e4))$par
    a <- soln3[1]; S12 <- soln3[2]
}

boot.par.init[i.sim,] <- c(a,alpha1,alpha2,mu1,mu2,S11,S22,S12,eta1,eta2)
```

```
est.mu1 <- mean(R1)
est.v1 <- var(R1)
est.s1 <- mean((R1-mean(R1))^3)
est.k1 <- mean((R1-mean(R1))^4)
est.mu2 <- mean(R2)
est.v2 <- var(R2)
est.s2 <- mean((R2-mean(R2))^3)
est.k2 <- mean((R2-mean(R2))^4)
est.cv <- cov(R1,R2)
est.ck <- mean((R1-mean(R1))^2*(R2-mean(R2))^2)
mom.eq <- function(x){
    x <- x/parscale
    a <- x[1]
    alpha1 <- x[2]
    alpha2 <- x[3]
    mu1 <- x[4]
    mu2 <- x[5]
    S11 <- x[6]
    S22 <- x[7]
    S12 <- x[8]
    eta1 <- x[9]
    eta2 <- x[10]
    if(!(0<a & a<min(1/alpha1,1/alpha2) & alpha1>0 & alpha2>0 & S11>0 & S22>0 &
    abs(S12/sqrt(abs(S11*S22)))<1)){
        return(Inf)
    }
    cokurt.formula <- 2*a^2*(min(alpha1,alpha2)^2*S12^2+2*alpha1*alpha2*min(alpha1,alpha2)*
    S12*mu1*mu2+alpha1^2*alpha2^2*mu1^2*mu2^2)+
    2*a*(min(alpha1,alpha2)^2*S12^2+4*alpha1*alpha2*min(alpha1,alpha2)*S12*mu1*mu2+3*
    alpha1^2*alpha2^2*mu1^2*mu2^2+
    alpha1^2*alpha2*S22*mu1^2+alpha1*alpha2^2*S11*mu2^2+0.5*alpha1*alpha2*S11*S22)+
    alpha1*alpha2*mu1^2*mu2^2+alpha1*S22*mu1^2+alpha2*S11*mu2^2+S11*S22
    component11 <- mu1+eta1 - est.mu1
    component21 <- S11+alpha1*mu1^2 - est.v1
    component31 <- 3*alpha1*S11*mu1+2*alpha1^2*mu1^3 - est.s1
    component41 <- 3*S11^2+3*alpha1*(S11^2+2*S11*mu1^2)+3*alpha1^2*(4*S11*mu1^2+mu1^4)+6*
    alpha1^3*mu1^4 - est.k1
    component12 <- mu2+eta2 - est.mu2
    component22 <- S22+alpha2*mu2^2 - est.v2
    component32 <- 3*alpha2*S22*mu2+2*alpha2^2*mu2^3 - est.s2
    component42 <- 3*S22^2+3*alpha2*(S22^2+2*S22*mu2^2)+3*alpha2^2*(4*S22*mu2^2+mu2^4)+6*
    alpha2^3*mu2^4 - est.k2
    component13 <- a*(min(alpha1,alpha2)*S12+alpha1*alpha2*mu1*mu2) - est.cv
    component23 <- cokurt.formula - est.ck
    rv <- sum(c(component11,component21,component31,component41,component12,component22,
    component32,component42,component13,component23)^2)
    return(rv)
}
# Estimate all parameters together
par <- c(a,alpha1,alpha2,mu1,mu2,S11,S22,S12,eta1,eta2)
parscale <- 1/c(scale,1/scale,1/scale,scale,scale,scale,scale,scale,scale,scale)
par.optim <- optim(par*parscale,mom.eq,method="Nelder-Mead",control=list(maxit=1e4))
a.mom <- a <- (par.optim$par/parscale)[1]
alpha1.mom <- alpha1 <- (par.optim$par/parscale)[2]
alpha2.mom <- alpha2 <- (par.optim$par/parscale)[3]
mu1.mom <- mu1 <- (par.optim$par/parscale)[4]
mu2.mom <- mu2 <- (par.optim$par/parscale)[5]
```

```
    S11.mom <- S11 <- (par.optim$par/parscale)[6]
    S22.mom <- S22 <- (par.optim$par/parscale)[7]
    S12.mom <- S12 <- (par.optim$par/parscale)[8]
    eta1.mom <- eta1 <- (par.optim$par/parscale)[9]
    eta2.mom <- eta2 <- (par.optim$par/parscale)[10]
    boot.par[i.sim,] <- c(a,alpha1,alpha2,mu1,mu2,S11,S22,S12,eta1,eta2)
}
```

Maximum Likelihood

```
marg.pdf.type <- "explicit"
n.sim <- 100
set.seed(46473448)
boot.par.init <- array(NA,c(n.sim,11))
boot.par <- array(NA,c(n.sim,10))
optim.list <- list()

tol.val <- 1e-4
for(i.sim in 1:n.sim){
    print(i.sim)
    return.process <- sim.vag(a.true,alpha1.true,alpha2.true,mu1.true,mu2.true,S11.true,
    S22.true,rho.true,t.max=1000)
    R1 <- return.process[,1]+eta1.true
    R2 <- return.process[,2]+eta2.true
    # Intial values from MOM
    a <- a.mom; alpha1 <- alpha1.mom; alpha2<- alpha2.mom; mu1 <- mu1.mom; mu2 <- mu2.mom;
    S11 <- S11.mom; S22 <- S22.mom; S12 <- S12.mom; eta1 <- eta1.mom; eta2 <- eta2.mom
    # MLE Marginal 1
    alphaVG.llf.mar1 <- function(par){
        par <- par/parscale
        alpha1 <- par[1]
        mu1 <- par[2]
        S11 <- par[3]
        eta1 <- par[4]
        if(!(alpha1>0 & S11>0)){
            return(Inf)
        }
        if(marg.pdf.type=="fourier"){
            v1 <- alpha1*mu1^2+S11
            Grid.n <- 2^11
            Grid.v <- 0.01*sqrt(v1)
            Grid.s <- 1/(Grid.n*Grid.v)
            Grid.x <- Grid.v[1]*((-Grid.n/2):(Grid.n/2-1))
            Grid.z <- 2*pi*Grid.s[1]*((-Grid.n/2):(Grid.n/2-1))
            Grid.x <-  Grid.x+eta1
            Z1 <- Grid.z
            sgn <- (-1)^(0:(Grid.n-1))
            charfn <- (1+alpha1*(-1i*mu1*Z1+S11*Z1^2/2))^(-1/alpha1*delta)*sgn
            P <- Re(fft(charfn))
            P <- P*sgn*Grid.s
            unit.area <- median(diff(Grid.x))
            P[P<0] <- 0
            P <- P/(sum(P)*unit.area)
            eval.pdf <- function(R.obs){
                x.ind.le <- which(pdf$xy<R.obs)
```

```
            if(length(x.ind.le)!=0){
                x.ind <- max(x.ind.le)
            }else{
                x.ind <- 1
            }
            rv <- pdf$pdf[x.ind]
            return(rv)
        }
        R.obs <- R1
        evaluated.pdf.val <- Vectorize(eval.pdf)(R.obs)
    }else if(marg.pdf.type=="explicit"){
        R.obs <- R1
        evaluated.pdf.val <- dvg(R.obs,vgC=eta1,sigma=sqrt(S11),theta=mu1,nu=alpha1)
    }
    log.lik.fn <- -sum(log(evaluated.pdf.val))
    return(log.lik.fn)
}
par <- c(alpha1,mu1,S11,eta1)
parscale <- 1/c(1/scale,scale,scale,scale)
if(is.finite(alphaVG.llf.mar1(par*parscale))){
    ok1 <- 1
}else{
    ok1 <- 2
    par <- par.before.all[c(2,4,6,9)]
}
if(marg.pdf.type=="fourier"){
    par.optim <- optim(par*parscale,alphaVG.llf.mar1,method="Nelder-Mead",
    control=list(maxit=1e4))
}else if(marg.pdf.type=="explicit"){
    par.optim <- optim(par*parscale,alphaVG.llf.mar1,method="Nelder-Mead",
    control=list(maxit=1e4))
    for(rep.optim in 1:3){
        par.optim <- optim(par.optim$par,alphaVG.llf.mar1,method="Nelder-Mead",
        control=list(maxit=1e4))
    }
}
alpha1 <- (par.optim$par/parscale)[1]
mu1 <- (par.optim$par/parscale)[2]
S11 <- (par.optim$par/parscale)[3]
eta1 <- (par.optim$par/parscale)[4]
d1 <- rbind(par,par.optim$par)
# MLE Marginal 2
alphaVG.llf.mar2 <- function(par){
    par <- par/parscale
    alpha2 <- par[1]
    mu2 <- par[2]
    S22 <- par[3]
    eta2 <- par[4]
    if(!(alpha2>0 & S22>0)){
        return(Inf)
    }
    if(marg.pdf.type=="fourier"){
        v2 <- alpha2*mu2^2+S22
        Grid.n <- 2^11
        Grid.v <- 0.01 *sqrt(v2)
        Grid.s <- 1/(Grid.n*Grid.v)
        Grid.x <- Grid.v[1]*((-Grid.n/2):(Grid.n/2-1))
```

```
        Grid.z <- 2*pi*Grid.s[1]*((-Grid.n/2):(Grid.n/2-1))
        Grid.x <-  Grid.x+eta2
        Z2 <- Grid.z
        sgn <- (-1)^(0:(Grid.n-1))
        charfn <- (1+alpha2*(-1i*mu2*Z2+S22*Z2^2/2))^(-1/alpha2*delta)*sgn
        P <- Re(fft(charfn))
        P <- P*sgn*Grid.s
        unit.area <- median(diff(Grid.x))
        P[P<0] <- 0
        P <- P/(sum(P)*unit.area)
        pdf <- list(xy=Grid.x,pdf=P)
        eval.pdf <- function(R.obs){
            x.ind.le <- which(pdf$xy<R.obs)
            if(length(x.ind.le)!=0){
                x.ind <- max(x.ind.le)
            }else{
                x.ind <- 1
            }
            rv <- pdf$pdf[x.ind]
            return(rv)
        }
        R.obs <- R2
        evaluated.pdf.val <- Vectorize(eval.pdf)(R.obs)
    }else if(marg.pdf.type=="explicit"){
        R.obs <- R2
        evaluated.pdf.val <- dvg(R.obs,vgC=eta2,sigma=sqrt(S22),theta=mu2,nu=alpha2)
    }
    log.lik.fn <- -sum(log(evaluated.pdf.val))
    return(log.lik.fn)
}
par <- c(alpha2,mu2,S22,eta2)
parscale <- 1/c(1/scale,scale,scale,scale)
if(is.finite(alphaVG.llf.mar2(par*parscale))){
    ok2 <- 1
}else{
    ok2 <- 2
    par <- par.before.all[c(3,5,7,10)]
}
if(marg.pdf.type=="fourier"){
    par.optim <- optim(par*parscale,alphaVG.llf.mar2,method="Nelder-Mead",
    control=list(maxit=1e4))
}else if(marg.pdf.type=="explicit"){
    par.optim <- optim(par*parscale,alphaVG.llf.mar2,method="Nelder-Mead",
    control=list(maxit=1e4))
    for(rep.optim in 1:3){
        par.optim <- optim(par.optim$par,alphaVG.llf.mar2,method="Nelder-Mead",
        control=list(maxit=1e4))
    }
}
alpha2 <- (par.optim$par/parscale)[1]
mu2 <- (par.optim$par/parscale)[2]
S22 <- (par.optim$par/parscale)[3]
eta2 <- (par.optim$par/parscale)[4]
d2 <- rbind(par,par.optim$par)
# Joint parameters
alphaVG.llf <- function(par){
    par <- par/parscale
```

```
if(length(par)==2){
    a <- par[1]
    rho <- par[2]
    S12 <- rho*sqrt(S11*S22)
}else{
    a <- par[1]
    alpha1 <- par[2]
    alpha2 <- par[3]
    mu1 <- par[4]
    mu2 <- par[5]
    S11 <- par[6]
    S22 <- par[7]
    S12 <- par[8]
    eta1 <- par[9]
    eta2 <- par[10]
}
v1 <- alpha1*mu1^2+S11
v2 <- alpha2*mu2^2+S22
Grid.n <- 2^11
Grid.v <- 0.01*sqrt(c(v1,v2))
Grid.s <- 1/(Grid.n*Grid.v)
Grid.x <- rbind(Grid.v[1]*((-Grid.n/2):(Grid.n/2-1)),Grid.v[2]*((-Grid.n/2):(Grid.n/2-1)))
Grid.z <- 2*pi*rbind(Grid.s[1]*((-Grid.n/2):(Grid.n/2-1)),Grid.s[2]*
((-Grid.n/2):(Grid.n/2-1)))
Grid.x <-  Grid.x+array(c(eta1,eta2),c(2,Grid.n))
Z1 <- array(Grid.z[1,],c(Grid.n,Grid.n))
Z2 <- t(array(Grid.z[2,],c(Grid.n,Grid.n)))
sgn <- 2*repmat(diag(2),Grid.n/2)-1
charfn <- (1+alpha1*(-1i*mu1*Z1+S11*Z1^2/2))^((a-1/alpha1)*delta)*
(1+alpha2*(-1i*mu2*Z2+S22*Z2^2/2))^((a-1/alpha2)*delta)*
(1+alpha1*(-1i*mu1*Z1+S11*Z1^2/2)+alpha2*(-1i*mu2*Z2+S22*Z2^2/2)+min(alpha1,alpha2)*
S12*Z1*Z2)^(-a*delta)*sgn
P <- Re(fft(charfn))
P <- P*sgn*Grid.s[1]*Grid.s[2]
unit.area <- median(diff(Grid.x[1,]))*median(diff(Grid.x[2,]))
P[P<0] <- 0
P <- P/(sum(P)*unit.area)
pdf <- list(xy=Grid.x,pdf=P)
eval.pdf <- function(R.obs){
    x.ind.le <- which(pdf$xy[1,]<R.obs[1])
    if(length(x.ind.le)!=0){
        x.ind <- max(x.ind.le)
    }else{
        x.ind <- 1
    }
    y.ind.le <- which(pdf$xy[2,]<R.obs[2])
    if(length(y.ind.le)!=0){
        y.ind <- max(y.ind.le)
    }else{
        y.ind <- 1
    }
    rv <- pdf$pdf[x.ind,y.ind]
    return(rv)
}
R.obs <- rbind(R1,R2)
R.obs.list <- split(R.obs,rep(1:ncol(R.obs), each = nrow(R.obs)))
evaluated.pdf.val <- unlist(lapply(R.obs.list,eval.pdf))
```

```
        log.lik.fn <- -sum(log(evaluated.pdf.val))
        return(log.lik.fn)
    }
    par <- c(a,S12/sqrt(S11*S22))
    parscale <- c(scale,scale)
    if(is.finite(alphaVG.llf(par*parscale))){
        ok3 <- 1
    }else{
        ok3 <- 2
        par <- c(par.before.all[1],par.before.all[8]/sqrt(par.before.all[6]*par.before.all[7]))
        if(is.finite(alphaVG.llf(par*parscale))){
        }else{
            ok3 <- 3
            par <- c(min(1/alpha1,1/alpha2)/2,0)
            if(is.finite(alphaVG.llf(par*parscale))){
            }else{
                ok3 <- 4
            }
        }
    }
    par.optim.sub <- optim(par*parscale,alphaVG.llf,method="Nelder-Mead",control=list(maxit=1e4))
    a <- (par.optim.sub$par/parscale)[1]
    rho <- (par.optim.sub$par/parscale)[2]
    S12 <- rho*sqrt(S11*S22)
    boot.par.init[i.sim,] <- c(a,alpha1,alpha2,mu1,mu2,S11,S22,S12,eta1,eta2,100*ok1+10*ok2+ok3)
    eps <- 1e-6
    if(a < eps){
        a <- eps
    }else if(a > min(1/alpha1,1/alpha2)-eps){
        a <- min(1/alpha1,1/alpha2)-eps
    }
    if(S12/sqrt(S11*S22) > 1-eps){
        rho <- 1-eps
        S12 <- rho*sqrt(S11*S22)
    }else if(S12/sqrt(S11*S22) < -1+eps){
        rho <- -1+eps
        S12 <- rho*sqrt(S11*S22)
    }
    # Estimate all parameters together
    par <- c(a,alpha1,alpha2,mu1,mu2,S11,S22,S12,eta1,eta2)
    parscale <- 1/c(scale,1/scale,1/scale,scale,scale,scale,scale,scale,scale,scale)
    par.optim <- optim(par*parscale,alphaVG.llf,method="Nelder-Mead",control=list(maxit=1e4))
    a <- (par.optim$par/parscale)[1]
    alpha1 <- (par.optim$par/parscale)[2]
    alpha2 <- (par.optim$par/parscale)[3]
    mu1 <- (par.optim$par/parscale)[4]
    mu2 <- (par.optim$par/parscale)[5]
    S11 <- (par.optim$par/parscale)[6]
    S22 <- (par.optim$par/parscale)[7]
    S12 <- (par.optim$par/parscale)[8]
    eta1 <- (par.optim$par/parscale)[9]
    eta2 <- (par.optim$par/parscale)[10]
    boot.par[i.sim,] <- c(a,alpha1,alpha2,mu1,mu2,S11,S22,S12,eta1,eta2)
    optim.list[[i.sim]] <- par.optim
}
```

Digital Moment Estimation

```
n <- length(R1)
q <- seq(0.05,0.95,len=10)
n.qu <- length(q)
k <- numeric(n.qu)
n.est <- 10000
make.plot <- FALSE
eps <- 1e-6
n.points <- 10
trans.mat <- diag(c(1,1))
dme.est <- function(type,i.sim=i.sim,data=NULL){
    if(type=="mse"){
        R1 <- all.R[[i.sim]][,1]
        R2 <- all.R[[i.sim]][,2]
    }else if(type=="est"){
        R1 <- data[,1]
        R2 <- data[,2]
    }
    k1 <- quantile(R1,prob=q)
    k2 <- quantile(R2,prob=q)
    # Marginal Parameters
    k <- k1
    q.tmp <- q
    for(i in 1:length(k)){
        q.tmp[i] <- sum(R1<=k[i])/n
    }
    solved <- optim(c(0,log(sqrt(var(R1))),log(1),0),obj.fn,control=list(maxit=1e4),q=q.tmp,k=k)
    marg.par.1 <- c(solved$par[1],exp(solved$par[2])^2,exp(solved$par[3]),solved$par[4])
    k <- k2
    q.tmp <- q
    for(i in 1:length(k)){
        q.tmp[i] <- sum(R2<=k[i])/n
    }
    solved <- optim(c(0,log(sqrt(var(R2))),log(1),0),obj.fn,control=list(maxit=1e4),q=q.tmp,k=k)
    marg.par.2 <- c(solved$par[1],exp(solved$par[2])^2,exp(solved$par[3]),solved$par[4])
    alpha1 <- alpha1.est <- marg.par.1[3]
    alpha2 <-   alpha2.est <- marg.par.2[3]
    mu1 <-  mu1.est <- marg.par.1[1]
    mu2 <-  mu2.est <- marg.par.2[1]
    S11 <- S11.est <- marg.par.1[2]
    S22 <-  S22.est <- marg.par.2[2]
    eta1 <- eta1.est <- marg.par.1[4]
    eta2 <- eta2.est <- marg.par.2[4]
    quan.q <- array(NA,c(length(k1),length(k2)))
    for(i in 1:length(k1)){
        for(j in 1:length(k2)){
            k.tmp <- trans.mat%*%c(k1[i],k2[j])
            quan.q[i,j] <- sum(R1<=k.tmp[1,] & R2<=k.tmp[2,])/n
        }
    }
    a.lowerbound <- 0; a.upperbound <- min(1/alpha1,1/alpha2)
    rho.lowerbound <- -1; rho.upperbound <- 1
    a.est.points <- seq(a.lowerbound+eps,a.upperbound-eps,len=n.points)
    rho.est.points <- seq(rho.lowerbound+eps,rho.upperbound-eps,len=n.points)
    par.surface <- outer(X=a.est.points,Y=rho.est.points,FUN=obj.fn.2d,
```

```
    alpha1.est=alpha1.est,alpha2.est=alpha2.est,
    mu1.est=mu1.est,mu2.est=mu2.est,S11.est=S11.est,S22.est=S22.est,
    eta1.est=eta1.est,eta2.est=eta2.est,quan.q=quan.q,k1=k1,k2=k2)
    grid <- expand.grid(a.est.points,rho.est.points)
    loess.df <- data.frame(surface=c(par.surface),grid1=grid[,1],grid2=grid[,2])
    l2 <- loess(surface~grid1+grid2,data=loess.df)
    l2.fit <- array(predict(l2),c(n.points,n.points))
    a.est <- grid[which(l2.fit==min(l2.fit)),1]
    rho.est <- grid[which(l2.fit==min(l2.fit)),2]
    loess.min <- optim(c(a.est,rho.est),get.to.min,method="L-BFGS-B",control=list(maxit=1e4),
    lower=c(a.lowerbound,rho.lowerbound)+eps,
    upper=c(a.upperbound,rho.upperbound)-eps,loess.fit=l2)
    a.est <- loess.min$par[1]
    rho.est <- loess.min$par[2]
    min.a.rho <- loess.min$value
    if(make.plot==TRUE){
        pmat <- persp(a.est.points,rho.est.points,l2.fit,
        theta=30, phi=20, ticktype='detailed',
        zlim=c(min(par.surface),max(par.surface)),
        xlab="a", ylab="rho", zlab="error",
        main=paste("a = ",round(a.est,4),", rho = ",round(rho.est,4),sep=""))
        points(trans3d(grid[,1],grid[,2],c(par.surface),pmat), pch=20)
        points(trans3d(a.est,rho.est,min.a.rho,pmat), pch=20,col="red",cex=2.5)
    }
    rm(R1,R2)
    return(c(marg.par.1,marg.par.2,c(a.est,rho.est)))
}

data <- cbind(R1,R2)
set.seed(46473448)
n.sim <- 100
all.R <- list()
for(i in 1:n.sim){
    return.process <- sim.vag(a.true,alpha1.true,alpha2.true,mu1.true,mu2.true,S11.true,S22.true,
    rho.true,t.max=1000)
    R1 <- return.process[,1]+eta1.true
    R2 <- return.process[,2]+eta2.true
    all.R[[i]] <- cbind(R1,R2)
}
set.seed(3444)
random.seeds <- sample(1:1e6,size=n.sim)
boot.par <- array(NA,c(n.sim,10))
tt <- proc.time()
for(i.sim in 1:n.sim){
    print(i.sim)
    set.seed(random.seeds[i.sim])
    boot.par[i.sim,] <- dme.est(type="mse",i.sim=i.sim)
}
```

Kolmogorov-Smirnov and Chi-Squared Statistics

```
# KS statistic
type.method <- "DME"
set.seed(46473448)
all.R <- list()
```

```
for(i in 1:n.sim){
    return.process <- sim.vag(a.true,alpha1.true,alpha2.true,mu1.true,mu2.true,S11.true,S22.true,
    rho.true,t.max=1000)
    R1 <- return.process[,1]+eta1.true
    R2 <- return.process[,2]+eta2.true
    all.R[[i]] <- cbind(R1,R2)
}
set.seed(83180)
gof.stats <- rep(NA,n.sim)
for(i in 1:n.sim){
    print(i)
    if(type.method %in% c("MLE","MOM")){
    a <-  boot.par[i,1]; alpha1 <- boot.par[i,2]; alpha2 <- boot.par[i,3]; mu1 <- boot.par[i,4]
    mu2 <- boot.par[i,5]; S11 <- boot.par[i,6]; S22 <- boot.par[i,7]; S12 <- boot.par[i,8]
    eta1 <- boot.par[i,9]; eta2 <- boot.par[i,10]; rho <- S12/sqrt(S11*S22)
}else if(type.method=="DME"){
    marg.par.1 <- boot.par[i,1:4]; marg.par.2 <- boot.par[i,5:8]; a <-  boot.par[i,9]
    rho <- boot.par[i,10]; alpha1 <- marg.par.1[3]; alpha2 <- marg.par.2[3];
    mu1 <- marg.par.1[1]; mu2 <- marg.par.2[1]; S11 <- marg.par.1[2]; S22 <- marg.par.2[2]
    eta1 <- marg.par.1[4]; eta2 <- marg.par.2[4]; S12 <- rho*sqrt(S11*S22)
}
return.process <- sim.vag(a,alpha1,alpha2,mu1,mu2,S11,S22,rho,t.max=1000)
R1.fit <- return.process[,1]+eta1
R2.fit <- return.process[,2]+eta2
ret.ori <- all.R[[i]]
ret.fit <- cbind(R1.fit,R2.fit)
gof.stats[i] <- peacock2(ret.fit,ret.ori)
}
mean(gof.stats)
round(mean(gof.stats),3)

# Chi-sq statistic
type.method <- "DME"
set.seed(46473448)
gof.stats <- rep(NA,n.sim)
for(i in 1:n.sim){
print(i)
if(type.method %in% c("MLE","MOM")){
a <-  boot.par[i,1]; alpha1 <- boot.par[i,2]; alpha2 <- boot.par[i,3]; mu1 <- boot.par[i,4]
mu2 <- boot.par[i,5]; S11 <- boot.par[i,6]; S22 <- boot.par[i,7]; S12 <- boot.par[i,8]
eta1 <- boot.par[i,9]; eta2 <- boot.par[i,10]; rho <- S12/sqrt(S11*S22)
}else if(type.method=="DME"){
marg.par.1 <- boot.par[i,1:4]; marg.par.2 <- boot.par[i,5:8]; a <-  boot.par[i,9]
rho <- boot.par[i,10]; alpha1 <- marg.par.1[3]; alpha2 <- marg.par.2[3];
mu1 <- marg.par.1[1]; mu2 <- marg.par.2[1]; S11 <- marg.par.1[2]; S22 <- marg.par.2[2]
eta1 <- marg.par.1[4]; eta2 <- marg.par.2[4]; S12 <- rho*sqrt(S11*S22)
}
x1 <- Grid.x[1,]
x2 <- Grid.x[2,]
joint <- P
n <- length(R1)
X1 <- R1
X2 <- R2
mg.cdf.1 <- function(x1){
rv <- try(pvg(x1, vgC=eta1, sigma=sqrt(S11), theta=mu1, nu=alpha1))
if(is.character(rv)==FALSE){
    return(rv)
```

```
}else{
    integrand <- function(x){
        dvg(x, vgC=eta1, sigma=sqrt(S11), theta=mu1, nu=alpha1)
    }
    rv <- try(integrate(integrand,lower=-Inf,upper=x1)$val)
    if(is.numeric(rv)){
        return(rv)
    }
}
}
mg.pdf.1 <- function(x1){
rv <- try(dvg(x1, vgC=eta1, sigma=sqrt(S11), theta=mu1, nu=alpha1))
if(is.numeric(rv)){
    return(rv)
}
}
pdf.1 <- mg.pdf.1(x1)
get.ind <- function(R.obs){
x.ind.le <- which(x1<R.obs[1])
if(length(x.ind.le)!=0){
    x.ind <- max(x.ind.le)
}else{
    x.ind <- 1
}
y.ind.le <- which(x2<R.obs[2])
if(length(y.ind.le)!=0){
    y.ind <- max(y.ind.le)
}else{
    y.ind <- 1
}
rv <- c(x.ind,y.ind)
return(rv)
}
rosenblatt.trans <- function(X1.val,X2.val){
X.ind <- get.ind(c(X1.val,X2.val))
Z1 <- mg.cdf.1(x1[X.ind[1]])
cond.dens <- joint[X.ind[1],1:X.ind[2]]/pdf.1[X.ind[1]]
cond.prob <- diff(x2)[1]*sum(cond.dens)
Z2 <- cond.prob
return(c(Z1,Z2))
}
rosenblatt.trans <- Vectorize(rosenblatt.trans)
unif.sq <- rosenblatt.trans(X1,X2)
xy <- t(unif.sq)
nbins <- 10
x.bin <- y.bin <- seq(1/nbins,1-1/nbins,length=nbins-1)
freq <-  as.data.frame(table(findInterval(xy[,1], x.bin),findInterval(xy[,2], y.bin)))
freq[,1] <- as.numeric(as.character(freq[,1]))+1
freq[,2] <- as.numeric(as.character(freq[,2]))+1
freq2D <- diag(nbins)*0
freq2D[cbind(freq[,1], freq[,2])] <- freq[,3]
obsv <- freq2D
expd <- n/nbins^2
chi.sq <- sum((obsv-expd)^2/expd)
1-pchisq(chi.sq,df=nbins^2-1)
gof.stats[i] <- chi.sq
}
```

```
mean(gof.stats)
round(mean(gof.stats),3)
```

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